ESSENTIAL MATHEMATICS

ADVANCED EDITION

Worksheets 1–34: Complete Course

Jurcone Ramiro

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Preface

This work synthesizes core mathematical concepts from the high school curriculum.

The motivation for creating this book was the recognized need for a single book containing mathematical formulas, solution methods and model solutions, for which it would otherwise be necessary to consult all high school textbooks and many problem collections.

The goal is to provide a succinct and focused presentation of the main concepts, without claiming to replace work from textbooks, exam problem books or specialized collections. Rather, it aims to be both an introduction to the study of high school mathematics and a reference manual for quick consultation of the main formulas and solution methods.

The topics covered are numbered according to the system Sheet 1, Sheet 2, Sheet 3, and so on.

In order to prepare for various exams, it is recommended that after completing the present material, preparation be done using a collection of exam subjects for the respective type of exam, with the aim of developing the specific skills for that type of examination.

As an example in this regard, exam subjects from previous years may be useful.

The Author

HOW TO USE THIS BOOK

Frequently Asked Questions

Q: Who is this book for?

A: This book serves multiple audiences:

a) Students seeking quick reference

- Formulas, methods, and solution strategies — all in one place
- Perfect for exam preparation and homework support
- No need to search through multiple textbooks

b) Educated parents helping their children

- Quickly recall formulas and methods you once knew
- Clear explanations you can pass on to your child
- Organized systematically for easy navigation

c) Hardworking students who want deep understanding

- Not just "how" but "WHY" — rigorous mathematical thinking
- Build solid foundations that last a lifetime
- Self-study optimized — learn independently with confidence

d) Diaspora families

- Available in both **Romanian** and **English** editions
- Preserve Romanian mathematical rigor while living abroad
- Help children excel in any educational system

Q: What is the mathematical level of this book?

A: This book covers the curriculum of a **good European high school** in a major city, with particular emphasis on Romanian standards.

Context: Romanian mathematics education is known internationally for its exceptional rigor. Students from Romanian high schools consistently excel at international competitions and top universities worldwide. This book reflects that tradition.

Difficulty rating: (4/5) — Intermediate to Advanced

- Starts accessible (basic concepts, trigonometry)
- Progresses to university level (linear algebra, calculus, integrals)
- Peaks with abstract mathematics (algebraic structures)

Comparison:

- More rigorous than typical Western high school
- Equivalent to first 1–2 years of technical university
- Similar to top preparatory programs

Note: Standards vary by country, but this book represents the **European rigorous tradition** where understanding comes before memorization.

Q: Can I print this book?

A: Yes! This book is formatted for standard **A4 paper**. You can print individual worksheets or the entire book on any home or office printer.

Many students find printed worksheets helpful for quick formula reference during homework or exam preparation. Keep them in a binder for easy access.

This book is your "samurai sword" — a tool of exceptional quality, given freely. What you achieve with it depends entirely on you.

Good luck!

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SHEET 1: Plane Geometry

Terms Used:

The following terms are considered to be known from middle school. Make sure, however, that they are 100% clear to you:

- line, ray, segment, perpendicular bisector of a segment.
- parallel lines, secant line, alternate interior angles, alternate exterior angles, corresponding angles
- angle bisector
- symmetric: symmetric of a point with respect to a line, symmetric of a line with respect to a line.
- angles: acute, obtuse, complementary, supplementary
- triangle: arbitrary triangle, isosceles, equilateral, right triangle, congruent triangles (cases of congruence),
- triangle: similar triangles (cases of similarity), Thales' theorem.
- triangle: sum of angles in a triangle, exterior angle of a triangle (what it is and how to calculate it),
- triangle: midline (what it is and how to calculate it)

Important Lines in a Triangle:

• Angle Bisector: Divides an angle into two equal parts. Any point on the bisector is equidistant from the sides of the angle. The angle bisectors of a triangle intersect at the center of the inscribed circle.

• Median: Connects the vertex of a triangle with the midpoint of the opposite side. The medians of a triangle intersect at the center of gravity G of the triangle. G is located two-thirds from the vertex and one-third from the base.

• Perpendicular Bisector: Is perpendicular to the midpoint of a segment. Any point on the perpendicular bisector is equidistant from the endpoints of the segment. The perpendicular bisectors of a triangle intersect at the center of the circumscribed circle of the triangle.

• Altitude: Is the perpendicular drawn from a vertex of the triangle to the opposite side. The altitudes of a triangle intersect at the orthocenter of the triangle ("Ortho" means right in Greek)

Area of an Arbitrary Triangle:

(1) S= $\frac{base*altitude}{2}$

(2) S= $\frac{b*c*sin(A)}{2}$

(3) S= $\frac{a*b*c}{4*R}$, where R= radius of the circumscribed circle (intersection of perpendicular bisectors)

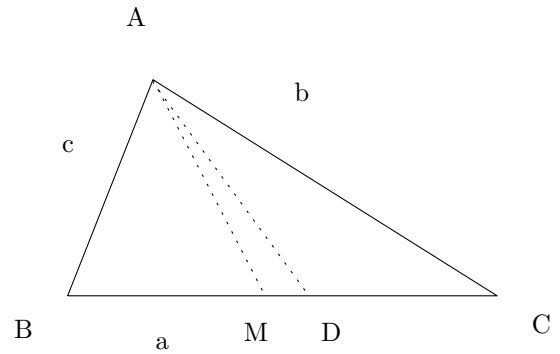
(4) $S = r * p$, where r= radius of the inscribed circle (intersection of angle bisectors), p= semiperimeter=$\frac{a+b+c}{2}$

(5) S= $\sqrt{p*(p-a)*(p-b)*(p-c)}$, where p=semiperimeter. (Heron's Formula)

(6) Useful idea: Expressing the area in two ways, in problems, and finding the unknown

Theorems in the Arbitrary Triangle

Let ABC be an arbitrary triangle, AD= median, AM= angle bisector



•Law of Sines: $\frac{a}{sin(A)} = \frac{b}{sin(B)} = \frac{c}{sin(C)} = 2 * R$, where R=radius of the circumscribed circle

•Law of Cosines (Generalized Pythagorean Theorem):

$a^2 = b^2 + c^2 - 2 * b * c * cos(A)$

Remarks:

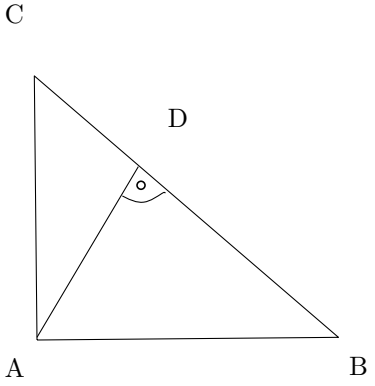
- For A=acute, cos(A) > 0, therefore $-2 * b * c * cos(A)$ is a negative quantity

- For A=obtuse, cos(A) < 0, therefore $-2 * b * c * cos(A)$ is a positive quantity

•Median Theorem: $AD^2 = \frac{2*(AB^2+AC^2)-BC^2}{4}$

•Angle Bisector Theorem: $\frac{BM}{MC} = \frac{AB}{AC}$

Right Triangle Let ABC be a right triangle, A= 90° , AD $\perp$ BC



• Formulas for sin, cos, tan, cot

• Values of sin, cos, tan, cot for 30, 60, 45 degrees

• B=90-C, therefore sin(B)=cos(C), cos(B)=sin(C), tan(B)=cot(C), cot(B)=tan(C), etc.

• Pythagorean Theorem: $BC^2 = AB^2 + AC^2$

• Altitude Theorem: $AD^2 = DB * DC$

• Leg Theorem: $AB^2 = BC * DB$, respectively $AC^2 = BC * DC$

- The leg opposite the 30-degree angle = half of the hypotenuse
- S= $\frac{leg*leg}{2}$

Polygons That Can Be Easily Decomposed into Triangles:

- Square
- Rectangle
- Parallelogram
- Trapezoid
- Rhombus

A useful exercise is the decomposition of the aforementioned figures into triangles and obtaining the formulas for perimeter and area, based on the formulas presented for triangles.

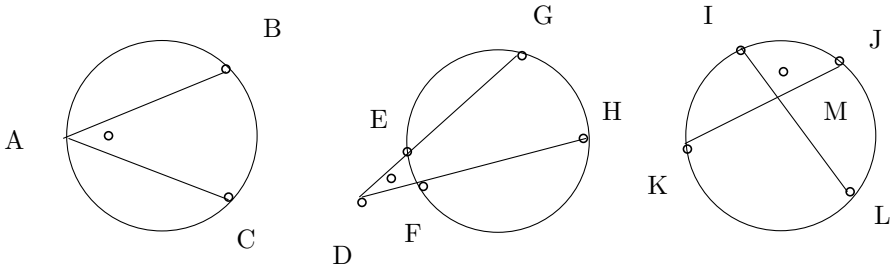
Circle

- Diameter=$2 * R$. Perimeter P=$2 * \pi * R$. Area S=$\pi * R^2$.
- Angle:

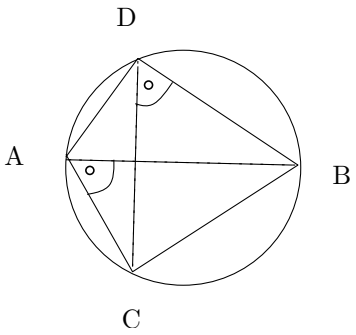
  a) on the circle: $\hat{A} = \widehat{CAB} = \frac{BC}{2}$

  b) exterior: $\hat{D} = \widehat{EDF} = \frac{GH-EF}{2}$

  c) interior: $\hat{M} = \widehat{KML} = \frac{IJ+KL}{2}$ according to the figures:



Cyclic Quadrilateral Let ABCD be a cyclic quadrilateral:



In a cyclic quadrilateral the following two properties are valid:

- The sum of opposite angles is 180 degrees, that is $\hat{A} + \hat{B} = 180$, respectively $\hat{C} + \hat{D} = 180$.

- The angle formed by one diagonal with one of the sides is equal to the angle formed by the other diagonal with the opposite side, that is for example $\widehat{CAB} = \widehat{CDB}$.

SHEET 2: Trigonometric Formulas:

$$sin(x) = \frac{opposite \quad leg}{hypotenuse}$$

$$cos(x) = \frac{adjacent \quad leg}{hypotenuse}$$

$$tg(x) = \frac{opposite \quad leg}{adjacent \quad leg}$$

$$ctg(x) = \frac{adjacent \quad leg}{opposite \quad leg}$$

$$tg(x) = \frac{sin(x)}{cos(x)}$$

$$ctg(x) = \frac{cos(x)}{sin(x)}$$

$$tg(x) = \frac{1}{ctg(x)}$$

$$ctg(x) = \frac{1}{tg(x)}$$

$sin(x) = cos(\frac{\pi}{2} - x)$ $\quad$ $cos(x) = sin(\frac{\pi}{2} - x)$ $\quad$ $tg(x) = ctg(\frac{\pi}{2} - x)$ $\quad$ $ctg(x) = tg(\frac{\pi}{2} - x)$

$$Fundamental\ formula\ of\ trigonometry:$$

$$sin^2x + cos^2x = 1$$

Useful: $cos^2(x) = \frac{1}{1+tg^2(x)}$ $\qquad$ $sin^2(x) = \frac{tg^2(x)}{1+tg^2(x)}$

If we denote $tg\frac{x}{2} = t$, then sin(x), cos(x), tg(x), ctg(x) can be expressed in terms of $tg\frac{x}{2}$ as follows:

$sin(x) = \frac{2t}{1+t^2}$ $\quad$ $cos(x) = \frac{1-t^2}{1+t^2}$ $\quad$ $tg(x) = \frac{sin(x)}{cos(x)} = \frac{2t}{1-t^2}$ $\quad$ $ctg(x) = \frac{cos(x)}{sin(x)} = \frac{1-t^2}{2t}$

$$sin(a+b) = sin(a) * cos(b) + cos(a) * sin(b)$$

$$sin(a-b) = sin(a) * cos(b) - cos(a) * sin(b)$$

$$cos(a+b) = cos(a) * cos(b) - sin(a) * sin(b)$$

$$cos(a-b) = cos(a) * cos(b) + sin(a) * sin(b)$$

$$sin(2a) = 2 * sin(a) * cos(a)$$

$$cos(2a) = cos^2(a) - sin^2(a)$$

$$cos(2a) = 2 * cos^2(a) - 1$$

$$cos(2a) = 1 - 2 * sin^2a$$

$$sin(3a) = sin(a+2a) = etc...$$

$$cos(3a) = cos(a+2a) = etc...$$

$$sin(\frac{a}{2}) = +- \sqrt{\frac{1-cos(a)}{2}}$$

$$cos(\frac{a}{2}) = +- \sqrt{\frac{1+cos(a)}{2}}$$

$$tg(\frac{a}{2}) = +- \sqrt{\frac{1-cos(a)}{1+cos(a)}}$$

$$ctg(\frac{a}{2}) = +- \sqrt{\frac{1+cos(a)}{1-cos(a)}}$$

$$tg(a+b) = \frac{tg(a)+tg(b)}{1-tg(a)*tg(b)}$$

$$tg(a-b) = \frac{tg(a)-tg(b)}{1+tg(a)*tg(b)}$$

$$ctg(a+b) = \frac{ctg(a)*ctg(b)-1}{ctg(a)+ctg(b)}$$

$$ctg(a-b) = \frac{1+ctg(a)*ctg(b)}{ctg(b)-ctg(a)}$$

$$tg(a+b+c) = tg([a+b]+c) = \frac{tg[a+b]+tg(c)}{1-tg[a+b]*tg(c)} = etc...$$

$$ctg(a+b+c) = ctg([a+b]+c) = \frac{ctg[a+b]*ctg(c)-1}{ctg[a+b]+ctg(c)} = etc...$$

$$sin(a)*cos(b) = \frac{sin(a+b)+sin(a-b)}{2}$$

$$cos(a)*cos(b) = \frac{cos(a+b)+cos(a-b)}{2}$$

$$cos(a)*sin(b) = \frac{sin(a+b)-sin(a-b)}{2}$$

$$sin(a)*sin(b) = \frac{cos(a-b)-cos(a+b)}{2}$$

$$sin(a)+sin(b) = 2*sin(\frac{a+b}{2})*cos(\frac{a-b}{2})$$

$$sin(a)-sin(b) = 2*cos(\frac{a+b}{2})*sin(\frac{a-b}{2})$$

$$cos(a)+cos(b) = 2*cos(\frac{a+b}{2})*cos(\frac{a-b}{2})$$

$$cos(a)-cos(b) = -2*sin(\frac{a+b}{2})*sin(\frac{a-b}{2})$$

The Trigonometric Circle

Write sin(x), cos(x), tg(x), ctg(x) for x=$30°, 45°, 60°$ and respectively $0°, 90°, 180°, 270°, (360)°$

SHEET 3: The Trigonometric Circle

Definition:
The trigonometric circle is a circle with radius R equal to 1 (equal to the unit)

Consequence 1: Expression of Angles in Radians

- Circle length = 2π R = 2π*1= 2π
- Circle length = traversing the entire circle, that is 360 degrees
- Therefore Circle length = 2π = 360
- Consequently: Since 360 degrees = 2π, therefore 180 degrees = π, it follows that 90 degrees = $\frac{\pi}{2}$ and 270 degrees = $\frac{3\pi}{2}$

For example we are interested in how many radians 45 degrees mean.
We start from:

180 degrees ..... π

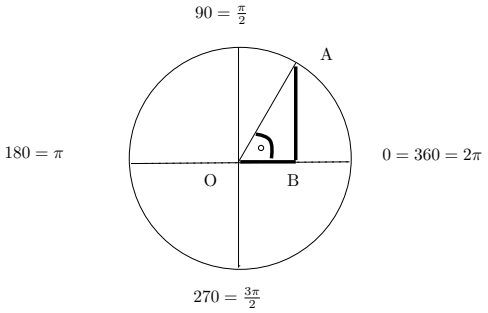
45 degrees (for example)..... x

We apply the rule of three to compute x.

Consequence 2: Finding sin, cos, tg, ctg for 0, 90, 180, 270, 360 degrees Consider the trigonometric circle below, therefore radius R=1.

sin(x)=$\frac{AB}{OA}$= $\frac{AB}{R=1}$ = AB, therefore the segment **AB represents sin(x)**

cos(x)=$\frac{OB}{OA}$= $\frac{OB}{R=1}$ =OB, therefore the segment **OB represents cos(x)**



1) For x=0 degrees, AB becomes 0 and OB becomes 1 (equal to radius R=1), therefore:
sin(0)=0, cos(0)=1, tg(0) = $\frac{0}{1}$=0, ctg(0)=$\frac{1}{0}$ =∞

2) For x=90 degrees, AB becomes 1 (equal to R=1) and OB becomes 0,therefore:
sin(90)=1, cos(90)=0, tg(90) = $\frac{1}{0}$=∞, ctg(90)=$\frac{0}{1}$ =0

3) For x=180 degrees, AB becomes 0 and OB becomes -1 (equal to radius R=1 in negative direction),therefore:
sin(180)=0, cos(180)=-1, tg(180) = $\frac{0}{-1}$=0, ctg(180)=$\frac{-1}{0}$ =-∞

4) For x=270 degrees, AB becomes -1 (equal to R=1 in negative direction) and OB becomes 0,therefore:
sin(270)=-1, cos(270)=0, tg(270) = $\frac{-1}{0}$=-∞, ctg(270)=$\frac{0}{-1}$ =0

Consequence 3: Values of sin, cos, tg, ctg for angles greater than 360 degrees

We consider known from middle school sin(x), cos(x), tg(x), ctg(x) for x=30, 60, 45 degrees:

For x=30: Since sin(x)=$\frac{1}{2}$ and cos(x)=$\frac{\sqrt{3}}{2}$, it follows that

tg=$\frac{sin(x)}{cos(x)}$ = $\frac{1}{\sqrt{3}}$ = $\frac{\sqrt{3}}{3}$ and ctg=$\frac{cos(x)}{sin(x)}$ =$\frac{\sqrt{3}}{1}$=$\sqrt{3}$

For x=60, since 60=90-30, sin(60)=cos(30), cos(60)=sin(30), tg(60)=ctg(30), ctg(60)= tg(30), that is for x=60:
sin(x)=$\frac{\sqrt{3}}{2}$, cos(x)=$\frac{1}{2}$, it follows that tg=$\frac{sin(x)}{cos(x)}$ = =$\sqrt{3}$ and ctg=$\frac{cos(x)}{sin(x)}$ =$\frac{1}{\sqrt{3}}$
= $\frac{\sqrt{3}}{3}$

For x=45, since the right triangle is isosceles (equal legs),
tg(x)=$\frac{opposite\ leg}{adjacent\ leg}$=1,
ctg(x)=$\frac{1}{tg(x)}$=1, the legs being equal and the hypotenuse being the same, sin(x)=cos(x) namely equal to $\frac{\sqrt{2}}{2}$

That is for x=45:
sin(x)=$\frac{\sqrt{2}}{2}$ cos(x)=$\frac{\sqrt{2}}{2}$, tg(x)=1 and ctg(x)=1

Example 1: Find sin(750)

Since 360 degrees means **one complete rotation,** , 750 degrees means 720(=360+360= two rotations and we arrive back at 0 degrees)+30.

Therefore sin(750) means that point A is in the same position as for 30 degrees.

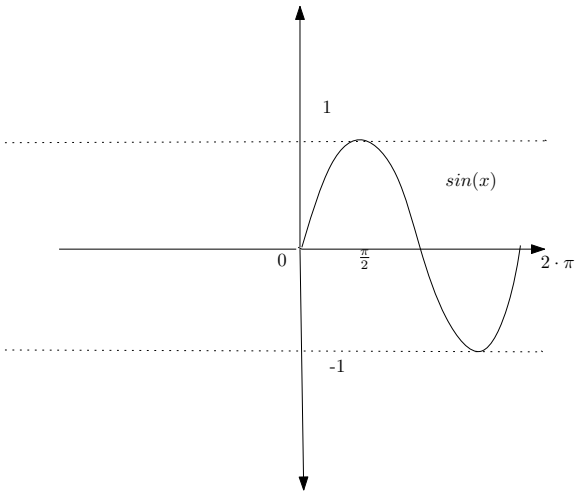
That is sin(750)=sin(30)=$\frac{1}{2}$.

Identically for cos(x), tg(x), ctg(x), that is cos(750)=cos(30), tg(750)=tg(30), ctg(750)=ctg(30).

Example 2: Find sin($\frac{13\pi}{4}$) First we transform $\frac{13\pi}{4}$ into degrees to judge more easily and obtain 585 degrees, that is 585=1* 360+225, meaning we have 1 complete rotation and then another 225 degrees. Therefore sin($\frac{13\pi}{4}$)= sin(225)= $-\frac{\sqrt{2}}{2}$. (quadrant 3, negative, equal to $-\sin(\pi/4)$)

SHEET 4: Graphs of Trigonometric Functions

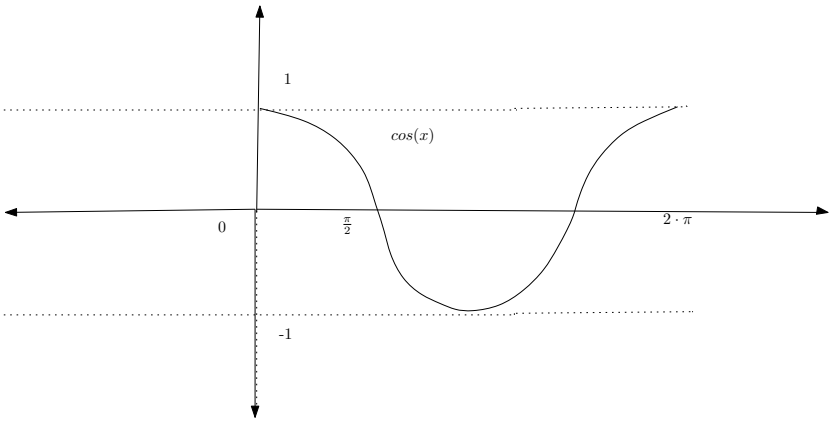
sin(x) : R → [−1, +1]



The function sin(x) has period $2 \cdot \pi$, is odd, is defined on R, takes values between -1 and +1 and is bijective for x $\in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Therefore the bijective restriction sin(x) : $[-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, +1]$ admits an inverse function, arcsin(x) : $[-1, +1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$, whose graph is symmetric with respect to sin(x) with respect to the first bisector, y=x.

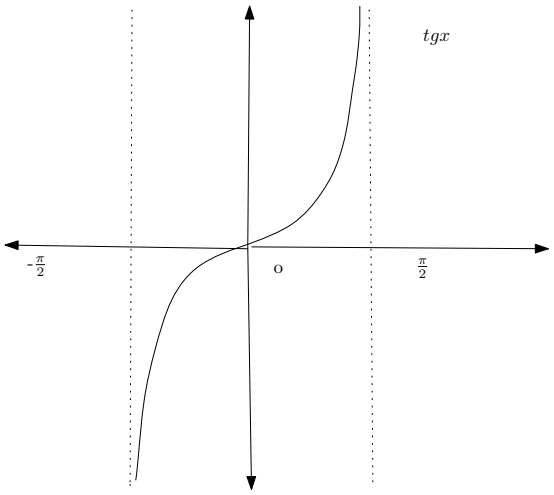
cos(x) : R → [−1, +1]



The function cos(x) has period $2 \cdot \pi$, is even, is defined on R, takes values between -1 and +1 and is bijective for x $\in [0, \pi]$.

Therefore the bijective restriction cos(x) : $[0, \pi] \to [-1, +1]$ admits an inverse function, arccos(x) : $[-1, +1] \to [0, \pi]$, whose graph is symmetric with respect to cos(x) with respect to the first bisector, y=x.

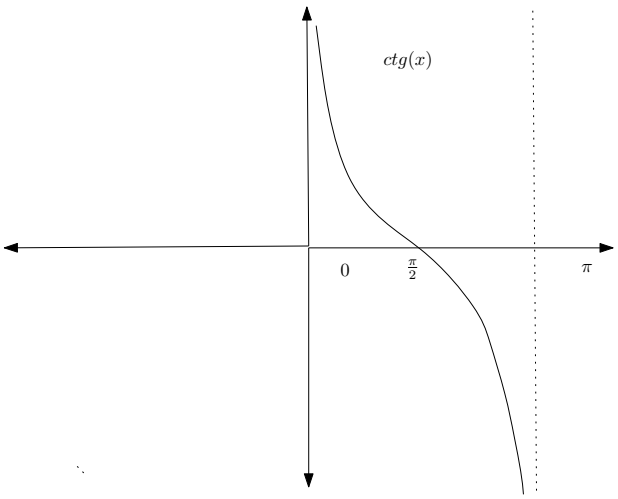
tg(x) : $\mathbf{R} \setminus \{ \frac{\pi}{2} + k \cdot \pi \} \to \mathbf{R}$



The function tg(x) has period π, is odd, is defined on
$\mathrm{R} \setminus \{ \frac{\pi}{2} + k \cdot \pi \}$, takes values between -$\infty$ and $+\infty$ and is bijective for x $\in (-\frac{\pi}{2}, +\frac{\pi}{2})$.

Therefore the bijective restriction tg(x) : $(-\frac{\pi}{2}, +\frac{\pi}{2}) \to (-\infty, +\infty)$ admits an inverse function, arctg(x) : $(-\infty, +\infty) \to (-\frac{\pi}{2}, +\frac{\pi}{2})$, whose graph is symmetric with respect to tg(x) with respect to the first bisector, y=x.

ctg(x) : $\mathbf{R} \setminus \{ k \cdot \pi \} \to \mathbf{R}$



The function ctg(x) has period π, is odd, is defined on
$\mathrm{R} \setminus \{ k \cdot \pi \}$, takes values between -$\infty$ and $+\infty$ and is bijective for x $\in (0, \pi)$.

Therefore the bijective restriction ctg(x) : $(0, \pi) \to (-\infty, +\infty)$ admits an inverse function, arcctg(x) : $(-\infty, +\infty) \to (0, \pi)$, whose graph is symmetric with respect to ctg(x) with respect to the first bisector, y=x.

SHEET 5: Trigonometric Equations and Inequalities

1. Elementary Trigonometric Equations

The following are considered known:

- The values for sin(x), cos(x), tg(x), ctg(x) are considered known, where x=30°, 45°, 60° and for 90°, 180°, 270°, as well as the method of transforming from degrees to radians.
- The trigonometric circle is considered known
- The graphs of the functions sin(x), cos(x), tg(x), ctg(x) are considered known
- It is considered known that sin(x), tg(x), ctg(x) are odd and cos(x) is even. It is also known that even functions are symmetric with respect to the y-axis, and odd functions are symmetric with respect to the origin
- It is considered known that sin(x) and cos(x) have period $2 \cdot \pi$ and tg(x) and ctg(x) have period π

An **elementary trigonometric equation** is considered to be an equation that has one of the following forms:

sin(x)=a , where a ∈ [-1,1]
cos(x)=b , where b ∈ [-1,1]
tg(x) =c , where c ∈ R
ctg(x)=d , where d ∈ R

The trap consists in the fact that solving these equations appears very simple, but in fact is complicated. For example for sin(x)=1/2, the first temptation is to quickly give the answer x= 30°. The answer is correct but incomplete, in the sense that there exist an infinity of solutions, namely: 30°, 150° (= 180-30) and for each of them, $+2\pi k$, where k ∈ Z namely for k>0 means rotations in the trigonometric direction and for k<0 means rotations in the inverse trigonometric direction. Writing the result in radians, the complete solution for the equation sin(x)=1/2 is x∈{ $\frac{\pi}{6}+2k\pi$} U { $\frac{5\cdot\pi}{6}+2k\pi$}

To avoid performing the previous reasoning each time, it is efficient to memorize the following formulas:

sin(x)=a	x=$(-1)^k\cdot$ arcsin(a) + kπ, for a ∈ [0,1]
sin(x)=a	x=$(-1)^{k+1}\cdot$ arcsin$\|a\|$+kπ, for a ∈ [-1,0)
cos(x)=b	x=± arccos(b)+2kπ, for b ∈ [0,1]
cos(x)=b	x=± arccos$\|b\|$+(2k+1)π, for b ∈ [-1,0]
tg(x)=c	x=arctg(c)+kπ
ctg(x)=d	x=arcctg(d)+kπ

The formulas for tg(x)=c and ctg(x)=d are easy to remember.
For sin(x)=a and cos(x)=b the formulas are more difficult to memorize, alternatively one can remember the method of deducing them, using the trigonometric circle.

2. Equations of the type sin(x)=sin(y)

For the following types:

sin(x)=sin(y)
cos(x)=cos(y)
sin(x)=cos(y)
tg(x) = tg(y)
ctg(x)=ctg(y)

To arrive at elementary trigonometric equations, it is recommended to transform the initial equation, for example:

For sin(x) = sin(y) we move everything to one side, sin(x)-sin(y)=0, transform into a product and obtain $2cos\frac{x+y}{2}sin\frac{x-y}{2} = 0$, that is $cos\frac{x+y}{2} = 0$ or $sin\frac{x-y}{2} = 0$, and we solve each equation separately.

For sin(x)=cos(y), we write sin(x)=sin($\frac{\pi}{2}$-y) and then proceed similarly to the previous problem.

3. Elementary Trigonometric Inequalities

An **elementary trigonometric inequality** is considered to be an inequality that has one of the following forms:

sin(x) > a
cos(x) > b
tg(x) > c
ctg(x) > d

The sign can be > , < , >= ,<= , etc.

To avoid memorizing other formulas, it is recommended to solve these inequalities as follows:

• For $sin(x) > a$ and $cos(x) > b$ we analyze the **trigonometric circle**, taking into account the rotations $2 \cdot k \cdot \pi$, where k ∈ Z.

• For $tg(x) > c$ and $ctg(x) > d$ we analyze the **graph of the function tg(x), respectively ctg(x)**, keeping in mind that tg(x) and ctg(x) have period π

4. ACTUAL Solution of Trigonometric Equations (respectively Trigonometric Inequalities)

In principle we follow the following steps:

1. We transform the given equation, respectively inequality, until we arrive at an **elementary equation**, respectively elementary inequality. The transformation is done using various techniques and the usual trigonometric formulas. Among these formulas, particularly useful are: the fundamental formula of trigonometry, the expression of sin(x), cos(x), tg(x), ctg(x) in terms of $tg\frac{x}{2}$, the expression of sin(x) and cos(x) in terms of tg(x), the half-angle formulas, which have radicals, in case the statement contains squared functions and we wish to eliminate squares, etc. Mastery of trigonometric formulas is recommended.

2. We solve the **elementary equation**, respectively elementary inequality, according to the methods presented earlier.

SHEET 6: Complex Numbers

A. Algebraic Form of Complex Numbers

Everything begins with the number $i = \sqrt{-1}$

• **Calculations** with **i**:

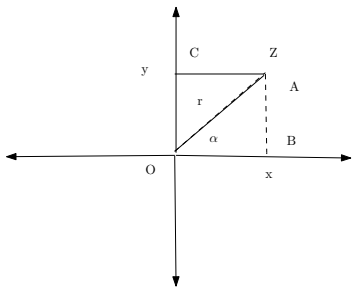
$i = \sqrt{-1}$
$i^2 = (\sqrt{-1})^2 = -1$
$i^3 = i^2 \cdot i = -1 \cdot i = -i$
$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = +1$
It is observed that i^4=1, therefore i to any power that is a multiple of 4 equals 1. This observation is useful for calculating expressions containing i^n.

For example $i^{30} = i^{28} \cdot i^2 = (i^4)^7 \cdot i^2 = 1 \cdot i^2 = -1$

• **New Terms** Let z=x+yi be a complex number (algebraic form).
-**x** is called the *real part* of the complex number and **y** is called the *imaginary part* of the complex number.
-**the conjugate of z** is denoted by $\bar{z}$ and is defined as $\bar{z}$=x-yi

B. Trigonometric Form of Complex Numbers

Let z=x+yi be a complex number (algebraic form).
Consider the following graphical representation:



We can write:
$sin(\alpha) = \frac{y}{r}$ therefore y=r·$sin(\alpha)$, respectively

$cos(\alpha) = \frac{x}{r}$ therefore x=r·$cos(\alpha)$

z=x+yi= r·$cos(\alpha)$+r·$sin(\alpha)$·i= $r[cos(\alpha) + i \cdot sin(\alpha)]$

• The form z=x+yi is called the *algebraic form of the complex number.* **x** is called the *real part* of the complex number and **y** is called the *imaginary part* of the complex number.

• The form z=$r[cos(\alpha) + i \cdot sin(\alpha)]$ is called the *trigonometric form of the complex number.* **r** is called the *modulus* of the complex number and α is called the *argument* of the complex number.

C. Writing a Complex Number in Trigonometric Form

Given a complex number z=x+yi. Write the number in trigonometric form, that is z=r(cosα +i sinα)

1. Determine the quadrant, based on the signs of x and y
2. Calculate $r = \sqrt{x^2 + y^2}$
3. Calculate $tg\alpha^* = \left|\frac{y}{x}\right|$ it follows that α^*= arctg$\left|\frac{y}{x}\right|$, always α^* being an angle from the first quadrant, because $\frac{y}{x}$ is taken in absolute value.
4. Calculate α based on α^* and based on the quadrant determined in step 1, as follows:

For quadrant 1, take $\alpha = \alpha^*$
For quadrant 2, take $\alpha = \pi - \alpha^*$
For quadrant 3, take $\alpha = \pi + \alpha^*$
For quadrant 4, take $\alpha = 2\pi - \alpha^*$

5. Write the complex number in trigonometric form,
that is z=r(cosα +i sinα)

D. Useful Formulas for Complex Numbers in Trigonometric Form

Let z_1=r_1(cosα_1 +i sinα_1) , z_2=r_2(cosα_2 +i sinα_2) , z=r(cosα +i sinα) ,

(1) *Multiplication:* $z_1 \cdot z_2 = r_1 \cdot r_2[cos(\alpha_1 + \alpha_2) + i \cdot sin(\alpha_1 + \alpha_2)]$

(2) *Division:* $\frac{z_1}{z_2} = \frac{r_1}{r_2}[cos(\alpha_1 - \alpha_2) + i \cdot sin(\alpha_1 - \alpha_2)]$

Since $z^n = z \cdot z \cdot z.... \cdot z = r \cdot r \cdot ...r[cos(\alpha + \alpha + \cdot\cdot + \alpha) + i \cdot sin(\alpha + \alpha \cdot\cdot + \alpha)]$

we obtain:

(3) *Power:* $z^n = r^n[cos(n \cdot \alpha) + i \cdot sin(n \cdot \alpha)]$

(4) *Radical:* $\sqrt[n]{z} = z^{\frac{1}{n}} = \sqrt[n]{r} \cdot (cos\frac{\alpha + 2k\pi}{n} + i \cdot sin\frac{\alpha + 2k\pi}{n})$

SHEET 7: Algebraic Formulas:

Formulas for $(a \pm b)^n$

$$(a+b)^2 = a^2 + 2*a*b + b^2$$

$$(a-b)^2 = a^2 - 2*a*b + b^2$$

$$(a+b)^3 = a^3 + 3*a^2*b + 3*a*b^2 + b^3$$

$$(a-b)^3 = a^3 - 3*a^2*b + 3*a*b^2 - b^3$$

$$(a-b)^4 = (a-b)^3*(a-b) \qquad etc...$$

Formulas for $a^n \pm b^n$

$$a^2 - b^2 = (a-b)*(a+b)$$

$$a^3 - b^3 = (a-b)*(a^2 + a*b + b^2)$$

$$a^3 + b^3 = (a+b)*(a^2 - a*b + b^2)$$

$$a^n - b^n = (a-b)*(a^{n-1} + a^{n-2}*b^1 + a^{n-3}*b^2 + ... + b^{n-1})$$

(for any natural number n)

$$a^n + b^n = (a+b)*(a^{n-1} - a^{n-2}*b^1 + a^{n-3}*b^2 - .... + b^{n-1})$$

(valid for n odd)

Sums

$S_1 = 1+2+3+\cdots+n = \frac{n*(n+1)}{2}$ S_1 is also denoted by $\sum_{k=1}^{n} k$

$S_2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n*(n+1)*(2*n+1)}{6}$ S_2 is also denoted by $\sum_{k=1}^{n} k^2$

$S_3 = 1^3+2^3+3^3+\cdots+n^3 = S_1^2 = (\frac{n*(n+1)}{2})^2$ S_3 is also denoted by $\sum_{k=1}^{n} k^3$

Example: Calculate the sum $S = 1*2 + 2*3 + ... + n*(n+1)$

Solution: S$= \sum_{k=1}^{n} k*(k+1) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k = S_2 + S_1 = \frac{n*(n+1)*(2*n+1)}{6} + \frac{n*(n+1)}{2}$

Powers and Radicals

$$(a * b)^m = a^m * b^m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

$$(a^m)^n = a^{m*n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$a^0 = 1$$

$$a^{-m} = \frac{1}{a^m}$$

$$\sqrt[m]{a^n} = a^{\frac{n}{m}}$$

Formula for Compound Radicals:
$\sqrt{A \pm \sqrt{B}} = \sqrt{\frac{A+C}{2}} \pm \sqrt{\frac{A-C}{2}}$, where $C = +\sqrt{A^2 - B}$.

Example: Calculate the expression E=$\sqrt{6 - \sqrt{11}}$.

Solution: I calculate C=$\sqrt{6^2 - 11} = \sqrt{25} = 5$, therefore E=$\sqrt{\frac{6+5}{2}} - \sqrt{\frac{6-5}{2}} = \sqrt{\frac{11}{2}} - \sqrt{\frac{1}{2}}$

SHEET 8: Quadratic Equation and Function

Second Degree Equation

$$a * x^2 + b * x + c = 0, \quad where \quad a \neq 0$$

- Roots $x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2*a}$ where $\Delta = b^2 - 4 * a * c$
- For $\Delta > 0$ there exist two real roots, x_1 *and* x_2 , with $x_1 \neq x_2$
- For $\Delta = 0$ there exist two real, equal and coinciding roots, $x_1 = x_2 = \frac{-b}{2*a}$
- For $\Delta < 0$ there do not exist real roots, or in other words, there exist two complex roots x_1 *and* x_2
- Viète's Relations: $S = x_1 + x_2 = \frac{-b}{a}$ $P = x_1 * x_2 = \frac{c}{a}$

- Finding the equation if the sum and product of the roots are known (for example we know that S=10 and P=20).
The equation is: $x^2 - S * x + P = 0$
For the previous example we obtain the equation: $x^2 - 10 * x + 20 = 0$

Second Degree Function

$$y = a * x^2 + b * x + c, \quad where \quad a \neq 0$$

- The graph of the second degree function is a parabola
- Sign of the second degree function:

Rule: *Between roots opposite sign to a, outside roots the sign of a, at the roots equal to zero.*

The application of the rule depends on Δ as follows:
- For $\Delta > 0$ it is exactly according to the rule.
- For $\Delta = 0$, there does NOT exist a zone between roots, therefore the function will be equal to zero at the roots, otherwise it will have the sign of a.
- For $\Delta < 0$, there exist neither roots nor the zone between roots, therefore the function will always have the sign of a.

- **Graphical representation of the second degree function.** Follow the steps:

1) If a > 0, the function has a minimum and if a < 0 it has a maximum.

2)For minimum and maximum the term "Vertex" or "extremum" is also used. This is a point let's say $V(x_v, y_v)$. It represents the lowest point of the graph in the case of the minimum, respectively the highest point in the case of the maximum. Calculate the coordinates of the vertex $V(x_v, y_v)$ with the formula $x_v = \frac{-b}{2*a}$ and $y_v = \frac{-\Delta}{4*a}$, so practically calculate the vertex $V(\frac{-b}{2*a}, \frac{-\Delta}{4*a})$.

3) Find the intersection with the Ox axis, making y=0 in the expression $y = a * x^2 + b * x + c$, that is practically solve the equation $a * x^2 + b * x + c = 0$. The values obtained for x represent the intersections with Ox. For example if we obtain two values x_1 and x_2, the intersection points with Ox will be $(x_1, 0)$ and $(x_2, 0)$.

4) Find the intersection with the Oy axis, making x=0 in the expression $y = a * x^2 + b * x + c$, that is we obtain $y = a * 0^2 + b * 0 + c$,

meaning always we actually obtain y=c. The point (0,c) represents the intersection of the graph with Oy.

5) Draw the graph using the previous data, in the following order: Draw the XOY axes, draw the vertex V, draw the form of the vertex (minimum or maximum), draw the intersections with Ox and the intersection with Oy and connect the points obtained. Keep in mind that the graph is symmetric with respect to the axis of symmetry passing through the vertex.

- **Monotonicity:**

- The function is decreasing from $-\infty$ up to x_{vertex} and increasing thereafter, from x_{vertex} up to $+\infty$ (if the Vertex is a minimum, that is for $a > 0$). Respectively,

- The function is increasing from $-\infty$ up to x_{vertex} and decreasing thereafter, from x_{vertex} up to $+\infty$ (if the Vertex is a maximum, that is for $a < 0$).

SHEET 9: Sign of the Quadratic Function

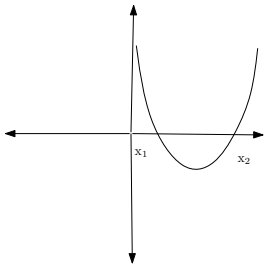
$y=ax^2+bx+c$, $a \neq 0$

The graph of the second degree function is a parabola. If a>0, the graph of the function has a minimum, and if a<0, the graph of the function has a maximum. The minimum, respectively maximum is called the vertex, respectively extreme point. The vertex has coordinates $V(\frac{-b}{2a}, \frac{-\Delta}{4a})$

There are three different situations, depending on whether $\Delta > 0$ or $\Delta = 0$ or $\Delta < 0$

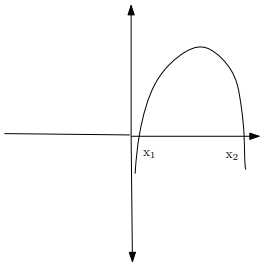
1. For $\Delta > 0$, there exist two real, different roots, $x_1 \neq x_2$, $x_1, x_2 \in R$.

The graph can be of the following type:



$\Delta > 0$ *si* $a > 0$

x	$-\infty$		x_1		x_2		$+\infty$
$y=ax^2+bx+c$	+	+	0	-	0	+	+



$\Delta > 0$ *si* $a < 0$

x	$-\infty$		x_1		x_2		$+\infty$
$y=ax^2+bx+x$	+	+	0	-	0	+	+

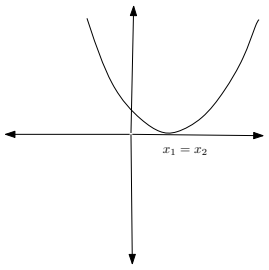
From the analysis of the graphs the following observations can be drawn:

• Between roots the second degree function has a sign opposite to a, and outside the roots it has the sign of a. At the roots, the function equals zero.

• For $\Delta > 0$, the function does not have a constant sign, being positive on certain intervals and negative on others.

2. For $\Delta = 0$, there exist two real, equal roots, $x_1 = x_2$, $x_1, x_2 \in R$.

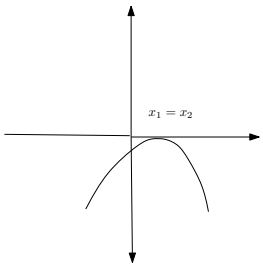
The graph can be of the following type:



$\Delta = 0$ *si* $a > 0$

x	$-\infty$		$x_1 = x_2$		$+\infty$
$y=ax^2+bx+c$	+	+	0	+	+

$y >= 0$



$\Delta = 0$ *si* $a < 0$

x	$-\infty$		$x_1 = x_2$		$+\infty$
$y=ax^2+bx+x$	-	-	0	-	-

$y <= 0$

From the analysis of the graphs the following observations can be drawn:

• The interval between roots *has disappeared, the roots being equal, respectively coinciding*, therefore the second degree function has the value zero at the roots and otherwise the sign of a.

• For $\Delta = 0$, the function **does not change sign** being positive or equal to zero for $a > 0$, respectively negative or equal to zero for $a < 0$.

• If in a problem it is necessary to impose the condition for $y >= 0$, we impose the conditions $\Delta <= 0$ and $a > 0$.

• If in a problem it is necessary to impose the condition for $y <= 0$, we impose the conditions $\Delta <= 0$ and $a < 0$.

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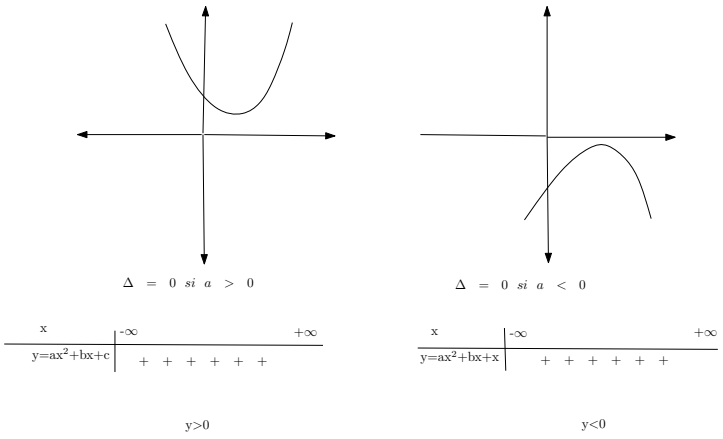
3. For $\Delta < 0$, there do not exist real roots

The graph can be of the following type:



From the analysis of the graphs the following observations can be drawn:

• The second degree function has the sign of a.

• For $\Delta < 0$, the function **does not change sign** being positive for $a > 0$, respectively negative for $a < 0$.

• If in a problem it is necessary to impose the condition for $y > 0$(strict), we impose the conditions $\Delta < 0$ and $a > 0$.

• If in a problem it is necessary to impose the condition for $y < 0$(strict), we impose the conditions $\Delta < 0$ and $a < 0$.

SHEET 10: Injective, Surjective, Bijective Functions

Injectivity

Definition Let f: A $\rightarrow$ B be a function. The function f is injective if for $x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$, for any $x_1, x_2 \in A$

Proving injectivity can be done by several methods, the choice of method depending on the type of function analyzed. For example:

1. If f(x) can be **represented graphically**, for f(x) to be injective, **any line parallel to Ox, drawn through B, must intersect the graph of the function at one point or none.** In other words, any line parallel to Ox through the codomain, must intersect the graph of the function at most at one point.

2. If f(x) can be **represented as a diagram**, usually for problems where the domain of definition A is a finite set, the function is not injective if there exist values in the codomain to which more than one element from the domain of definition corresponds.

3. If f(x) cannot be represented either graphically or as a diagram, one can try **by algebraic calculation**, that is we consider $x_1 \neq x_2$ and try to show that $f(x_1) \neq f(x_2)$. Practically we start from the difference $f(x_1) - f(x_2)$ and through various calculations we try to show that this difference is different from zero. Alternatively we can start from the ratio $\frac{f(x_1)}{f(x_2)}$ and try to demonstrate that this ratio is different from 1.

4. Injectivity can also be verified using material from **mathematical analysis**, studied in the 11th grade. Practically, for the function f(x) we calculate the derivative f'(x) and if we show that f'(x) is always positive on the domain of definition, it means that the function f(x) is increasing. If the function is increasing, it automatically means that the function is injective, because $x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$.

Similarly, if instead we obtain f'(x) always negative on the domain of definition, it means that the function f(x) is decreasing. If the function is decreasing, it automatically means that the function is injective, because $x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$.

It is observed that we used the property that a monotonic function (regardless of whether it is increasing or decreasing) is injective. The study of the monotonicity of the function was performed through the study of the sign of the derivative of the function. The fact that if f'>0 means that f=increasing, respectively if f'<0 means that f=decreasing, represents consequences of Lagrange's theorem.

Surjectivity

Definition Let f: A $\rightarrow$ B be a function. The function f is surjective if for y $\in$ B, there exists x $\in$ A, such that y=f(x).

Proving surjectivity can be done by several methods, the choice of method depending on the type of function analyzed. For example:

1. If f(x) can be **represented graphically**, for f(x) to be surjective, **any line parallel to Ox, drawn through B, must intersect the graph of the function at one point or more.** In other words, any line parallel to Ox through the codomain, must intersect the graph of the function at least at one point.

2. If f(x) can be **represented as a diagram**, usually for problems where the domain of definition A is a finite set, the function is not surjective if there exist values in the codomain to which **no element from the domain of definition corresponds.**

3. If f(x) cannot be represented either graphically or as a diagram, one can try **by algebraic calculation**, that is **denote f(x) by y and then solve for x as a function of y.** From this expression of x as a function of y, **analyze whether for any y from the codomain, there exists a corresponding x from the domain of definition.**
For example for f:R $\rightarrow$ R, f(x)=x^2+1, we make y=x^2+1 and solve for x=$\pm\sqrt{y-1}$. For the radical to exist we need y-1 >=0, that is y>=1, that is y $\in$ [1,∞). The codomain being R, means that for y $\in$ (-∞,1) there does not exist a corresponding x $\in$ R such that y=f(x), therefore the function is not surjective.

4. Surjectivity can also be verified using material from **mathematical analysis**, studied in the 11th grade. The notion of continuity of a function is used and Darboux's property. For example for f(x)=2^x+3^x, since the function is continuous, it means it has Darboux's property, therefore it is surjective. We recall **Darboux's property:**

A function has Darboux's property if it transforms an interval into another interval. That is, for f: I $\rightarrow$ J, we say that f has Darboux's property if for the domain of definition I being an interval, we obtain through the function f, the codomain J also as an interval. Continuous functions have Darboux's property. Therefore if I=interval, establishing surjectivity is reduced to studying the continuity of the function and invoking the implication of continuity on Darboux's property.

Bijectivity

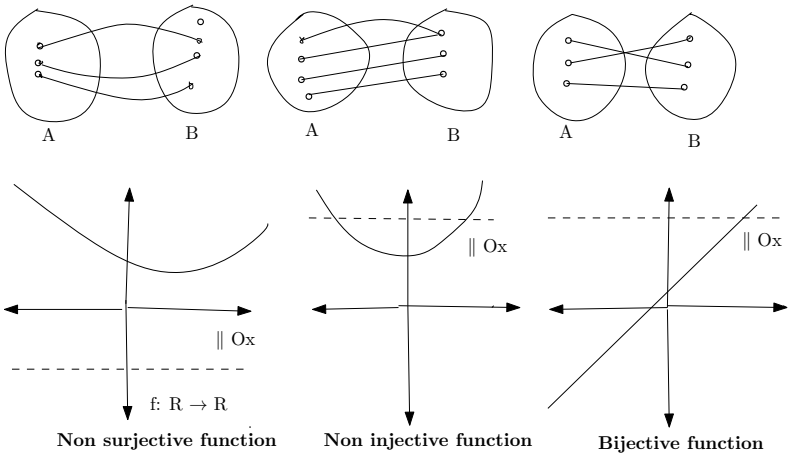
Definition Let f: A $\rightarrow$ B be a function. The function f is bijective if it is both injective and surjective.

The study of bijectivity is reduced to verifying, in turn, injectivity and surjectivity according to the methods presented earlier. If the function is both injective and surjective, we conclude that the function is bijective, otherwise we conclude that the function is not bijective.

The following observations can be made:

1. In the case where **graphical representation is used,** for a function to be bijective, that is both injective and surjective, **any line parallel to Ox, drawn through B, must intersect the graph of the function exactly at one point**.

2. In the case where **diagram representation is used,** for a function to be bijective, that is both injective and surjective, in the diagram there must be a 1 to 1 correspondence, that is to any element x $\in$ A there corresponds exactly one element y $\in$ B and vice versa. In other words, there must be a **1 to 1 correspondence**, which is also called **one-to-one correspondence**. **Examples:**



SHEET 11: Arithmetic Progressions. Geometric Progressions

Arithmetic Progressions

• Definition: Arithmetic progression = a sequence $a_1, a_2, ..., a_n$ with the property that each term equals the previous term plus the common difference (denoted r). In other words, $a_2 - a_1 = r$, $a_3 - a_2 = r$, etc. More precisely, $a_n - a_{n-1} = r$ for $n >= 2$. That is an A.P. is a sequence $a_1, a_2, ..., a_n$ with the property that

$$\mathbf{a_n = a_{n-1} + r}$$

This is formula 1. The common difference can be either positive or negative.

It is necessary that n be >= 2, because a_1 has no previous term.

• Formula 2 = Expression of a_n as a function of the first term (a_1) and common difference (r).

Since $a_1 = a_1$
Since $a_2 = a_1 + r$
Since $a_3 = a_2 + r = a_1 + 2 * r$
Since $a_4 = a_3 + r = a_1 + 3 * r$, etc., we obtain the formula:

$$\mathbf{a_n = a_1 + (n - 1) * r}$$

• Formula 3 = Verification if three numbers are in arithmetic progression

To verify if the numbers A, B, C are in A.P., we verify if the condition is satisfied:

$$\mathbf{B = \frac{A + C}{2}}$$

that is if the middle one is the arithmetic mean of the neighbors.

It is natural for it to be so, because A= B - r, C= B+r, therefore $A+C = 2 * B$

• Formula 4 = Sum of an arithmetic progression

For the arithmetic progression $a_1, a_2, ..., a_n$, the sum

$$\mathbf{S = a_1 + a_2 + ... + a_n = \frac{(a_1 + a_n) * n}{2}}$$

It is easily remembered by the formulation $\mathbf{S = \frac{(FirstTerm + LastTerm) * n}{2}}$

The proof is simple, namely:
S=$a_1+a_2+...+a_n = a_1+(a_1+r)+(a_1+2*r)+(a_1+3*r)+....(a_1+(n-1)*r)$
$= (a_1*n+r+2*r+3*r+...(n-1)*r) = n*a_1+r*(1+2+...+(n-1)) =$
$= n * a_1 + r * \frac{(n-1)(n-1+1)}{2} = n * a_1 + \frac{r*n*(n-1)}{2} = \frac{2*a_1+(n-1)*r}{2} * n =$
$\frac{a_1+a_1+(n-1)*r}{2} * n$
Therefore S=$\frac{(a_1+a_n)*n}{2}$

• Formula 5 = Sum of an arithmetic progression, alternative formula

If we express $a_n = a_1 + (n - 1) * r$ in the previous formula, we obtain another formula for S, namely:

$$\mathbf{S = \frac{(2 * a_1 + (n - 1) * r) * n}{2}}$$

Recommendation: If in the problem the first term and last term are known, formula 4 is useful, and if the first term and common difference are known, formula 5 is useful. In both situations we need to know the number of terms of the progression, denoted by n.

Geometric Progressions

• Definition: Geometric progression = a sequence $b_1, b_2, ..., b_n$ with the property that each term equals the previous term multiplied by the common ratio (denoted q). In other words, $b_2 = b_1 * q$, $b_3 = b_2 * q$, etc. More precisely, $b_n = b_{n-1} * q$ for $n >= 2$. That is a G.P. is a sequence $b_1, b_2, ..., b_n$ with the property that

$$\mathbf{b_n = b_{n-1} * q}$$

This is formula 1. The common ratio can be either positive or negative. It is necessary that n be >= 2, because b_1 has no previous term.

• Formula 2 = Expression of b_n as a function of the first term (b_1) and common ratio (q).
Since $b_1 = b_1$
Since $b_2 = b_1 * q$
Since $b_3 = b_2 * q = b_1 * q^2$
Since $b_4 = b_3 * q = b_1 * q^3$, etc., we obtain the formula:

$$\mathbf{b_n = b_1 * q^{n-1}}$$

• Formula 3 = Verification if three numbers are in geometric progression
To verify if the numbers A, B, C are in G.P., we verify if the condition is satisfied:

$$\mathbf{B^2 = A * C}$$

(the general case), respectively $B = \sqrt{A \cdot C}$ for $A, B, C > 0$,
that is if the middle one is the geometric mean of the neighbors.
It is natural for it to be so, because $A = \frac{B}{q}$, $C = B * q$, therefore $A * C = B^2$

• Formula 4 = Sum of a geometric progression
For the geometric progression $b_1, b_2, ..., b_n$, the sum

$$\mathbf{S = b_1 + b_2 + ... + b_n = b_1 * (\frac{q^n - 1}{q - 1})}$$

The proof is simple, namely:
S=$b_1 + b_2 + ... + b_n = b_1 + (b_1 * q) + (b_1 * q^2) + (b_1 * q^3) + ...(b_1 * q^{n-1})$
$= b_1(1 + q + q^2 + q^3 + ... + q^{n-1}) = b_1 * (\frac{q^n-1}{q-1})$ Q.E.D.

We used the formula $x^n - 1^n = (x-1) * (x^{n-1} + x^{n-2} + x^{n-3} + ... + x + 1)$ where in the role of x we used q and we extracted $(1+x+x^2+...+x^{n-1}) = \frac{x^n-1}{x-1}$

SHEET 12: Exponential Function

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y=a^x, a >0 , a$\neq$ 1

There are two different situations, depending on whether a>1 or a$\in$(0,1)

Justification of the fact that a >0 , a$\neq$ 1

The value of **a cannot have** the following values:

-It should not be that a equals 0, because 0 to any power equals 0.

-It should not be that a equals 1, because 1 to any power equals 1.

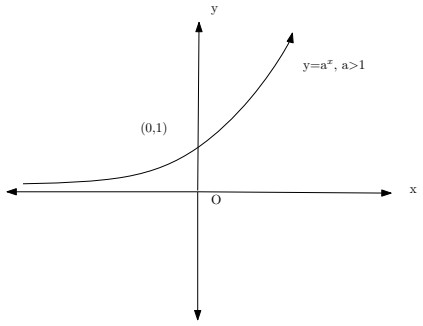
-It should not be that a is negative. For example if we have a=-3 and x=$\frac{1}{2}$, we would obtain a^x = (-3)$^{\frac{1}{2}}$, that is a^x = $\sqrt{-3}$, that is the square root of a negative number, which does not belong to R.

From the values a $\in$ R, if we eliminate a=0, a=1 and a<0, we obtain two permitted intervals, namely a $\in$ (0,1) and a $\in$ (1,∞), or in other words, a>0 and a $\neq$1

Case 1. a>1 For example let y=2^x. We draw the graph of the function by points, giving values to x:

x	...	-3	-2	-1	0	1	2	3	....
y		1/8	1/4	1/2	1	2	4	8	

We obtain the following graph:



From the analysis of the graph the following observations can be drawn:

• The function y=a^x, a> 1 is defined on R and takes values in (0,∞), that is
f: R $\rightarrow$ (0,∞), therefore the domain of definition is R, and the codomain is (0,∞).
It is observed that any line parallel to the Ox axis, drawn through the codomain intersects the graph of the function at exactly one point, therefore the function is bijective.
Being bijective, it means that it is both surjective and injective.
Due to the fact that it is injective, that is if f(x_1)=f(x_2) it means that x_1=x_2, in the case where we obtain an equation of the form $a^{f(x)}$=$a^{g(x)}$, we can conclude that f(x)=g(x).

• $a^x > 0$ for any x $\in$ R.
Therefore if we obtain an equation for example 5^x=-25 it means that the equation has no solutions, because 5^x is always positive.

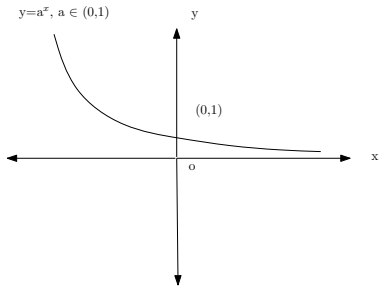
• The function a^x, for a> 1 is **increasing**, that is if $x_1 < x_2$ it means that $f(x_1) < f(x_2)$ and vice versa. If we obtain an inequality, for example 2^x <8, we can conclude that x< 3.

• Since a^0=1, the graph of the function y=a^x passes through the point (0,1) for any a> 1

Case 2. **a∈(0,1)** For example let y=$(\frac{1}{2})^x$. We draw the graph of the function by points, giving values to x:

x	...	-3	-2	-1	0	1	2	3 ....
y		8	4	2	1	1/2	1/4	1/8

We obtain the following graph:



From the analysis of the graph the following observations can be drawn:

• The function y=a^x, a∈ (0,1) is defined on R and takes values in (0,∞), that is
f: R → (0,∞), therefore the domain of definition is R, and the codomain is (0,∞). It is observed that any line parallel to the Ox axis, drawn through the codomain intersects the graph of the function at exactly one point, therefore the function is bijective. Being bijective, it means that it is both surjective and injective. Due to the fact that it is injective, that is if f(x_1)=f(x_2) it means that x_1=x_2, in the case where we obtain an equation of the form a$^{f(x)}$=a$^{g(x)}$, we can conclude that f(x)=g(x).

• a$^x > 0$ for any x ∈ R. Therefore if we obtain an equation for example $(\frac{1}{5})^x$=-$(\frac{1}{25})$ it means that the equation has no solutions, because $(\frac{1}{5})^x$ is always positive.

• The function a^x, for a∈ (0,1) is **decreasing**, that is if $x_1 < x_2$ it means that $f(x_1) > f(x_2)$ and vice versa. If we obtain an inequality for example $(\frac{1}{2})^x < \frac{1}{8}$, we can conclude that x> 3.

• Since a^0=1, the graph of the function y=a^x passes through the point (0,1) for any a∈ (0,1).

• **Practically for solving exponential equations** we try to arrive through various techniques at an equation of the form a$^{f(x)}$=a$^{g(x)}$, from which we conclude that f(x)=g(x).

• **Practically for solving exponential inequalities** we try to arrive through various techniques at an inequality of the form a$^{f(x)}$ < a$^{g(x)}$, which is interpreted according to the value of a, namely if a> 1 we preserve the sign of the inequality between f(x) and g(x), respectively if a∈ (0,1), we reverse the sign of the inequality between f(x) and g(x).

• It is useful to recall the formulas for powers
$(a^m)^n = a^{m*n}$, $\frac{a^m}{a^n} = a^{m-n}$, a^0=1, a$^{-m} = \frac{1}{a^m}$, $\sqrt[m]{a^n} = a^{\frac{n}{m}}$

• **The exponential function** y=a^x : R → (0,∞) being bijective, admits an inverse function namely **the logarithmic function** y=$\log_a^x$: (0,∞) → R, whose graph is symmetric with respect to y=a^x with respect to the first bisector, y=x. Of course two situations are generated for $\log_a^x$ namely for a>1 and for a∈(0,1).

SHEET 13: Logarithms

Formulas:

1) Definition: log_a^X is a number N, with the property that,

$$\text{if } \ log_a^X = N, \text{ then } X = a^N$$

For example $log_{10}^{100} = 2$ because $100 = 10^2$

2) $log_a^A + log_a^B = log_a^{A*B}$

3) $log_a^A - log_a^B = log_a^{\frac{A}{B}}$

4) $log_a^{A^m} = m * log_a^A$

5) $log_a^a = 1$

6) $log_{a=old}^X = \frac{log_{b=new}^X}{log_{b=new}^{a=old}}$

7) $log_{a^m}^X = \frac{1}{m} * log_a^X$

8) $log_a \sqrt[m]{X^n} = log_a^{X^{\frac{n}{m}}} = \frac{n}{m} * log_a^X$

Domain of definition:

For $log_a A$ it is necessary that

- A >0 $\rightarrow$ we obtain for example x belongs to interval I_1
- a >0 and a $\neq$1 $\rightarrow$ we obtain for example x belongs to interval I_2

The domain of definition is I_1 intersected with I_2

Base of the logarithm:

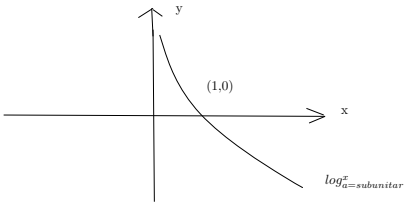
- lg^X means log_{10}^X that is *common logarithm*
- ln^X means log_e^X that is *natural logarithm*, where e≈2.718

Subunitary base or superunitary base.

Due to the condition from the *domain of definition*, for log_a^X, since we need $a > 0$ and $a \neq 1$, practically there are two situations for the base of the logarithm:

-CASE 1: a between (0,1). In this case, the logarithm function is decreasing, for example if we know that a is between (0,1) and we obtain in a problem $log_a^A < log_a^B$, we can draw the conclusion that $A > B$.

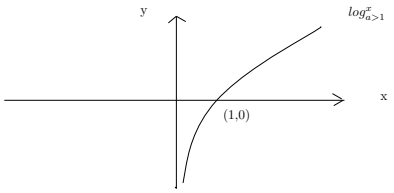
The graph of the logarithm function, for **subunitary** base is the following:



The following can be observed from the graph:

- $log_a^X : (0, \infty) \to R$
- the function is decreasing
- $log_a^1 = 0$
- the function is positive for X between (0,1)
- the function is zero for X=1
- the function is negative for X>1.

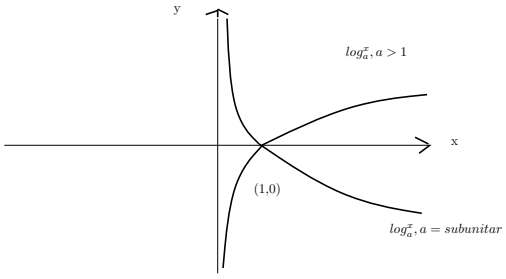
- **CASE 2: a between (1,∞)** In this case, the logarithm function is increasing, for example if we know that a is between (1,∞) and we obtain in a problem $log_a^A < log_a^B$, we can draw the conclusion that $A < B$. The graph of the logarithm function, for **superunitary** base is the following:



The following can be observed from the graph:

- $log_a^X : (0, \infty) \to R$
- the function is increasing
- $log_a^1 = 0$
- the function is negative for X between (0,1)
- the function is zero for X=1
- the function is positive for X>1.

Both cases can be synthesized in a single graph:



SHEET 14: Combinatorial Analysis

Factorial, Permutations, Arrangements, Combinations

Factorial

• By definition **n factorial** is denoted by **n!** and has the following calculation formula:

$$n! = 1 \cdot 2 \cdot 3 \cdot .... \cdot (n-1) \cdot n$$

• Examples:

a) 3!=$1 \cdot 2 \cdot 3 = 6$

b) 1!=1

c) 0!=1 (surprising at first sight, will be explained later)

• The following formulas result:

n!=$1 \cdot 2 \cdot 3 \cdot .... \cdot (n-1) \cdot n$, therefore it can also be written as follows:

n!=(n-1)!$\cdot n$

n!=(n-2)!$\cdot (n-1) \cdot n$

n!=(n-3)!$\cdot (n-2) \cdot (n-1) \cdot n$, etc.

Permutations

• **Permutations of n elements** is denoted by P_n and has the following calculation formula:

$$P_n = n!$$

therefore $P_n = n! = 1 \cdot 2 \cdot 3 \cdot .... \cdot (n-1) \cdot n$

• Example:

a) $P_3 = 3! = 1 \cdot 2 \cdot 3 = 6$

• What does "Permutations of n elements" mean

If we have for example a number of n=3 elements $\{a, b, c\}$, P_3 shows us **in how many ways** we can "permute (= exchange among themselves the 3 elements)", so as to form "teams" that:

a) contain all the n elements (both a and b and c)
b) no element repeats.

Practically we can form the following "teams":

{abc} {acb} {bac} {bca} {cab} {cba}

If we count the "previous teams" we see that there are 6. Indeed 6= $P_3 = 3! = 1 \cdot 2 \cdot 3$. In conclusion with P_n we can find **the number of "teams"** that can be generated, so it tells us **how many teams there are, not which teams these are**.

Arrangements

• **Arrangements of n elements taken k at a time** is denoted by A_n^k and has the following calculation formula:

$A_n^k = n \cdot (n-1) \cdot (n-2) \cdot .... \cdot (n-k+1)$

• Example:
$A_{10}^4 = 10 \cdot 9...(10-4+1) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$

• What A_n^k means:

Example 1: For example if we have 30 students and we want to group them 2 at a time in desks (order matters — like when you take a class picture), the number of these groupings is given by A_{30}^2, namely according to the previous formula we can calculate $A_{30}^2 = 30 \cdot 29 = 870$ that is they can be "arranged" in 870 ways.

Example 2: If we have n=3 elements for example the set $\{a, b, c\}$ and we want to "arrange" these elements in groups of 2, we obtain:

{ab} {ac} {bc} {ba} {cb} {ca} We observe that there are 6 groupings, that is $A_3^2 = 3 \cdot 2 = 6$. It is observed that both the grouping $\{ab\}$ and the grouping $\{ba\}$ appear that is **the order of the elements matters.**

Practically A_n^k is similar to P_n with the difference that in the "team" not all the n elements enter but only k (where k $\leq$ n).

• Domain of definition:

For A_n^k we need n, k $\in$ N and n $\geq$ k

One can imagine the absurdity of situations for n $<$ k or for n, k non-natural (for example negative, fractional, etc.)

Combinations

• **Combinations of n elements taken k at a time** is denoted by C_n^k and has the following calculation formula:

$C_n^k = \frac{A_n^k}{P_k}$

• **Practically** The formula $C_n^k = \frac{n!}{k!(n-k)!}$ is used

• Example:
$C_{10}^4 = \frac{10!}{4! \cdot 6!} = \frac{6! \cdot 7 \cdot 8 \cdot 9 \cdot 10}{6! \cdot 1 \cdot 2 \cdot 3 \cdot 4} = 210$

• What C_n^k means:

Example: If we have n=3 elements for example the set $\{a, b, c\}$ and we want to "combine" these elements in groups of 2, we obtain:

{ab} {ac} {bc} We observe that there are 3 groupings, that is $C_3^2 = 3$. It is observed that the grouping $\{ab\}$ appears **but not the grouping** $\{ba\}$ that is **the order of the elements does not matter.**

Practically C_n^k is similar to A_n^k with the difference that in the "team" the order of the elements does not matter, that is {a,b } is considered identical to {b,a }, so it is "counted" only once . It is also observed from the definition formula that $C_n^k \leq A_n^k$

• Domain of definition:

For C_n^k we need n, k $\in$ N and n $\geq$ k, as in the case of arrangements.

SHEET 15: Newton's Binomial

1. Newton's Binomial Formula

It is observed that the following formulas have similar structure:

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, etc.

In general for $(a+b)^n$ the following are observed:

There are n+1 terms, the first term being a^n, and the last b^n. More precisely the first is $a^n b^0$ and the last $a^0 \cdot b^n$, in general we have terms of the form $a^{n-k}b^k$, where k takes values from 0 to n. These terms are each multiplied by a coefficient. These coefficients have the formula C_n^k.

The grouping $(a+b)^n$ being formed of two terms, is called a binomial. After the name of the person who discovered the following formula, this is called **Newton's binomial**:

$$(a+b)^n = C_n^0 \cdot a^{n-0} \cdot b^0 + C_n^1 \cdot a^{n-1} \cdot b^1 + C_n^2 \cdot a^{n-2} \cdot b^2 + \cdots C_n^n \cdot a^{n-n} \cdot b^n$$

The following observations can be made:

- There are n+1 terms, because k takes values between 0 and n.

- The first term is actually equal to a^n, because C_n^0=1 and b^0=1.

- The last term is actually equal to b^n, because C_n^n=1 and a^0=1.

- The coefficients C_n^0, C_n^1, $\cdots$ C_n^n are called **binomial coefficients**.

- **Equidistant binomial coefficients are equal**, because it can be easily demonstrated that $C_n^k = C_n^{n-k}$.

- Newton's binomial can be written concisely in the following form:

$$(a+b)^n = \sum_{k=0}^{n} C_n^k \cdot a^{n-k} \cdot b^k$$

therefore as a sum having the general term

$$T_{k+1} = C_n^k \cdot a^{n-k} \cdot b^k$$

It is observed that the first term T_1 has k=0, the second term T_2 has k=1, etc. so indeed the notation of T_{k+1} for a value k is justified.

- If the binomial is a difference instead of a sum:

$$(a-b)^n = (a+[-b])^n = C_n^0 \cdot a^{n-0} \cdot (-b)^0 + C_n^1 \cdot a^{n-1} \cdot (-b)^1 + C_n^2 \cdot a^{n-2} \cdot (-b)^2 + \cdots C_n^n \cdot a^{n-n} \cdot (-b)^n$$

that is there is an alternation of signs starting with + then - and so on.

$$(a-b)^n = (a+[-b])^n = C_n^0 \cdot a^{n-0} \cdot b^0 - C_n^1 \cdot a^{n-1} \cdot b^1 + C_n^2 \cdot a^{n-2} \cdot b^2 + \cdots C_n^n \cdot a^{n-n} \cdot (-b)^n$$

- Now we can write Newton's binomial formula in the most general way:

$$(a+b)^n = \sum_{k=0}^{n} C_n^k \cdot a^{n-k} \cdot b^k$$

$$(a-b)^n = \sum_{k=0}^{n} (-1)^k \cdot C_n^k \cdot a^{n-k} \cdot b^k$$

therefore for $(a-b)^n$ the general term is

$$T_{k+1} = (-1)^k \cdot C_n^k \cdot a^{n-k} \cdot b^k$$

---

2. Useful Formulas

• Most problems with Newton's binomial are solved **starting from the formula**

$$T_{k+1} = C_n^k \cdot a^{n-k} \cdot b^k$$

for $(a+b)^n$

• The following two binomials are developed with Newton's binomial:

$(1+1)^n = 2^n = C_n^0 + C_n^1 + C_n^2 \cdots C_n^n \qquad (1)$

$(1-1)^n = 0 \ = C_n^0 - C_n^1 + C_n^2 - \cdots + (-1)^n C_n^n \qquad (2)$

Adding relation (1) to relation (2) we obtain: $C_n^0 + C_n^2 + C_n^4 \cdot \cdot = 2^{n-1}$

Subtracting relations (1) - (2) we obtain: $C_n^1 + C_n^3 + C_n^5 \cdots = 2^{n-1}$

In conclusion **the sum of even binomial coefficients equals the sum of odd binomial coefficients and equals** 2^{n-1}:

$$C_n^0 + C_n^2 + C_n^4 \cdot \cdot = C_n^1 + C_n^3 + C_n^5 \cdot \cdot = 2^{n-1}$$

• For problems where it is required to determine the rank of the largest term in the expansion, we start from the ratio $\frac{T_{k+1}}{T_{k+2}}$ which is compared with 1. That is we start from $\frac{T_{k+1}}{T_{k+2}} > 1$ and from this relation we solve for k. Then we interpret k and find the maximum term. For a binomial $(a+b)^n$, the ratio

$$\frac{T_{k+1}}{T_{k+2}} = \frac{k+1}{n-k} \cdot \frac{a}{b}$$

The formula being difficult to remember, it is recommended to demonstrate ad-hoc, starting from the ratio $\frac{T_{k+1}}{T_{k+2}}$ and expressing the terms T_{k+1} and T_{k+2} with the usual formula $T_{k+1} = C_n^k \cdot a^{n-k} \cdot b^k$ for $(a+b)^n$

Examples: Find the rank of the largest term in the expansion:

$$a)\ (1+0.1)^{100} \qquad b)\ (\frac{1}{2}+\frac{1}{2})^{100} \qquad c)\ (\frac{3}{4}+\frac{1}{4})^{100}$$

SHEET 16: Polynomials

1. Division of Polynomials

• **Method 1** The classical method of dividing two polynomials f(x) and g(x), according to the method studied in the 8th grade. It has the advantage that the divisor g(x) can be of any degree. It is recommended to perform the verification according to the formula:

$$\textbf{Dividend} = \textbf{Divisor} \times \textbf{Quotient} + \textbf{Remainder}$$

• **Method 2** Using Horner's scheme for dividing two polynomials f(x) and g(x), where g(x) is of the form (x-a). It has the disadvantage that g(x) must be of degree I. If g(x) is of higher degree, we decompose g(x) into factors of degree I, for example x^2-25 =(x-5)(x-(-5)) and we divide successively f by (x-5) and then the quotient obtained is divided by x-(-5). Horner's scheme has the advantage that it lends itself to computer processing.

2. Divisibility of Polynomials

To determine the *the greatest common divisor* **(gcd)** of two polynomials, the **Euclidean algorithm** is used, namely:

We divide f(x) by g(x) and obtain a quotient and a remainder. Then we divide the dividend by the obtained remainder. We continue performing divisions of the dividend by the remainder, until we obtain remainder=0. **The last non-zero remainder is gcd(f,g).**

Note 1 If *the least common multiple* **lcm(f,g)** is required, we use the property that

$$\textbf{lcm(f,g)} \cdot \textbf{gcd(f,g)} = \textbf{f} \cdot \textbf{g}$$

Practically, we calculate f ·g and then we calculate gcd(f,g) with the Euclidean algorithm. Then we find lcm(f,g)=$\frac{\text{f·g}}{\text{gcd(f,g)}}$

Note 2 If it is required to verify whether f and g are **relatively prime**, we use the property that **two polynomials are relatively prime if their gcd=1.**

Practically, we calculate gcd(f,g) with the Euclidean algorithm and if we obtain gcd(f,g)=1 we draw the conclusion that f and g are relatively prime, otherwise we draw the conclusion that they are not relatively prime.

3. Roots of Higher Degree Equations. Multiple Roots.

• A polynomial P_n of degree n, having roots $x_1, x_2, \cdots x_n$ can be written in the form:

$$P_n = (x - x_1) \cdot (x - x_2) \cdots (x - x_n)$$

• If a polynomial P(X) admits root x=a, then P(a)=0 (Bézout's Theorem).

• Multiple roots. An equation can have multiple roots (double, triple, etc.), in general of order k of multiplicity.

For example x=α is a double root if x_1=x_2=α. In this case P(x) appears in the form:

$$P(x)_n = (x - \alpha)^2 \cdot (x - x_3) \cdots (x - x_n)$$

It means that P(x) is exactly divisible by (x-α)^2, that is it is divided for example with Horner's scheme by (x-α) and then the obtained quotient is divided also exactly by (x-α). Another approach to the problem is to divide P(x) by (x-a)^2, that is by x^2-2ax+a^2 by the classical method of polynomial division and impose the condition that the remainder is zero.

The most effective for multiple roots, is to use the following theory:

A polynomial P(x) has root x=α as a multiple root of order k of multiplicity, if:

$$P(\alpha) = 0$$
$$P^{'}(\alpha) = 0$$
$$P^{''}(\alpha) = 0$$
$$\cdots$$
$$P^{k-1}(\alpha) = 0$$
$$P^{k}(\alpha) \neq 0$$

---

4. Viète's Relations

For example let the 3rd degree equation: $ax^3+bx^2+cx+d=0$ having roots x$_1$,x$_2$,x$_3$.

The following sums, called Viète's relations, can be written:

$$\begin{cases} S_1 = & x_1+x_2+x_3 & = -\frac{b}{a} \\ S_2 = & x_1\cdot x_2+x_1\cdot x_3+x_2\cdot x_3 & = +\frac{c}{a} \\ S_3 = & x_1\cdot x_2\cdot x_3 & = -\frac{d}{a} \end{cases}$$

Similarly Viète's relations can be written for any degree n.

If the roots of an equation are known, for example y$_1$,y$_2$,y$_3$ $\cdots$ y$_n$, and we wish to find the equation that has those roots, we calculate Viète's sums S$_1$,S$_2$, $\cdots$ S$_n$, then we write the expression of the equation that has those roots:

$$1\cdot Y^n - S_1\cdot Y^{n-1} + S_2\cdot Y^{n-2}\cdots S_n = 0$$

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5. Advanced Solutions of Higher Degree Equations

In addition to the classical methods of solving higher degree equations, advanced methods that use derivatives can also be employed, for example:

- Using **Rolle's Sequence**.
- Using the graphical representation of the functions that form the equation.

SHEET 17: Higher Degree Equations

Part I

Type 1. Biquadratic Equations

Example: Solve the equation $x^4 - 6x^2 + 6 = 0$

Idea: Denote $x^2 = y$ and obtain a second degree equation which is solved, then find x_1, x_2, x_3, x_4.

Type 2. Reciprocal Equations of Degree Three

Example: Solve the equation $5x^3 + 31x^2 + 31x^1 + 5 = 0$

Theory 1: A **reciprocal equation** is an equation which has equidistant coefficients that are equal.

Theory 2: Any reciprocal equation of odd degree admits the root x=-1.

Theory 3: A polynomial P_n of degree n, having roots $x_1, x_2, \cdots x_n$ can be written in the form:

$$P_n = (x - x_1) \cdot (x - x_2) \cdots (x - x_n)$$

Theory 4: If we have for a division Dividend, Divisor, Quotient, Remainder, the following relation is correct:

$$Dividend = Quotient \cdot Divisor + Remainder$$

Idea: Let $P_3(x)$ be the expression equal to zero. Since it admits the root x=-1, it means that $P_3(x)$ is divisible by x-(-1) that is by x+1. We perform the division of $P_3(x)$ by (x+1) and obtain $Quotient_2(x)$ and remainder=0. Therefore $P_3(x) = (x + 1) \cdot Quotient_2(x)$. We solve $Quotient_2(x) = 0$ and find the other two roots.

Type 3. Reciprocal Equations of Degree Four

Example: Solve the equation $2x^4 + 7x^3 + 9x^2 + 7x^1 + 2 = 0$

Idea: Divide the equation by x^2, after which denote $(x + \frac{1}{x}) = y$. Express everything in terms of y. Solve the second degree equation in y, then find x_1, x_2, x_3, x_4.

Type 4. Reciprocal Equations of Degree Five

Example: Solve the equation $20x^5 - 81x^4 + 62x^3 + 62x^2 - 81x^1 + 20 = 0$

Idea: Being a reciprocal equation of odd degree, it has the root x=-1. We proceed as in the case of reciprocal equations of degree three and from $P_5(x) = (x + 1)Q_4(x)$, by dividing $P_5(x)$ by (x+1) we obtain $Q_4(x)$ as a reciprocal equation of degree 4, which is solved like any reciprocal equation of degree four.

Type 5. Equations that Admit Root $x = a + b \cdot i$

Example 1: Solve the equation $ax^4 + bx^3 + cx^2 + dx^1 + e = 0$ knowing that it admits the root x=1+i.

Theory: If an equation admits the root x=a+bi, then it also admits the root x=a-bi

Idea: Let for example the polynomial $P_4(x)$ which we know admits the root $x_1 = a + bi$. According to the theory, $x_2 = a - bi$. This means that we can write

$$P_4(x) = (x - x_1) \cdot (x - x_2) \cdot Q_2(x)$$

We calculate $(x - x_1) \cdot (x - x_2)$ and eliminate i, we denote the obtained form for convenience of writing with R(x). Therefore $P_4(x) = R(x) \cdot Q_2(x)$. We find $Q_2(x)$ by dividing $P_4(x)$ by $R(x)$. We solve $Q_2(x) = 0$ and find from here x_3, x_4. We know x_1 from the statement as being x=a+bi, and x_2 we know from theory as being x=a-bi.

Example 2: Determine a and b, then solve the equation

$$x^4 - 7x^3 + 21x^2 + ax^1 + b = 0$$

knowing that it admits the root x=1+2i

Theory: If a polynomial P(X) admits the root x=a, then P(a)=0 (Bézout's Theorem).

Idea: Since P(x) admits the roots x=a and x=b, according to Bézout's theorem, we can write that P(a)=0 and P(b)=0. We obtained a system of two equations with two unknowns, which we solve and find a and b. Now we know the form of P(x) and we use the theory according to which if the polynomial admits the root x=1+2i, it means it also admits the root x=1-2i. We proceed as in the case of the previous problem and find the other roots.

Type 6. Equations that Admit Root $x = a + \sqrt{b}$

Example: Solve the equation $x^4 - 4x^3 + x^2 + 6x^1 + 2 = 0$ knowing that it admits the root $x = 1 - \sqrt{2}$.

Theory: If an equation admits the root $x = a + \sqrt{b}$, then it also admits the root $x = a - \sqrt{b}$

Idea: Similar to the previous problem. Let for example the polynomial $P_4(x)$ which we know admits the root $x_1 = a + \sqrt{b}$. According to the theory, $x_2 = a - \sqrt{b}$. This means that we can write

$$P_4(x) = (x - x_1) \cdot (x - x_2) \cdot Q_2(x)$$

We calculate $(x - x_1) \cdot (x - x_2)$ and eliminate $\sqrt{b}$, we denote the obtained form for convenience of writing with R(x). Therefore $P_4(x) = R(x) \cdot Q_2(x)$. We find $Q_2(x)$ by dividing $P_4(x)$ by $R(x)$. We solve $Q_2(x) = 0$ and find from here x_3, x_4. We know x_1 from the statement as being $x = a + \sqrt{b}$, and x_2 we know from theory as being $x = a - \sqrt{b}$.

SHEET 18: Higher Degree Equations

Part II

Type 7. Equations with Maximum Degree Coefficient =1

Example: Solve the equation $x^4 - 2x^3 - 5x^2 + 8x + 4 = 0$

Theory: The integer roots of the equation **could be** among the divisors of the constant term.

Idea: Extract the divisors of the constant term, e.g.: +1,-1,+2,-2,+4,-4 and verify in turn if they are roots with Bézout's theorem. That is, verify if P(+1)=0. If it equals zero, it means it is a root, otherwise it is not. Make verifications for all divisors of the constant term. If we find for example two roots, let x_1 and x_2, we write $P(x) = (x - x_1)(x - x_2)Q_2(x)$. We calculate $(x - x_1)(x - x_2)$ and obtain a second degree equation, we denote it for convenience of writing with R(x). We divide P(x) by R(x) and obtain $Q_2(x)$. We solve $Q_2(x) = 0$ and find x_3 and x_4.

Type 8. General Case = Equations with Maximum Degree Coefficient Different from 1

(Also valid for maximum degree coefficient equal to 1 as a particular case)

Example: Solve the equation $6x^4 - 17x^3 - x^2 + 8x - 2 = 0$

Theory: The roots of the equation **could be** of the form $\alpha = \frac{p}{q}$, p=divisor of the constant term, and q=divisor of the maximum degree coefficient.

Idea: The method involves many calculations. Extract the divisors of the constant term, e.g.: p= +1,-1,+2,-2 and the divisors of the maximum rank coefficient e.g.: q=+1,-1,+2,-2,+3,-3,+6,-6. Form all combinations of type $\alpha = \frac{p}{q}$, namely $\frac{+1}{+1}, \frac{+1}{-1}, \frac{+1}{+2} \cdot \cdot$, etc. and verify with Bézout's theorem if $P(\alpha)$=0. If two solutions are found, proceed further as in the case of the previous problem.

Type 9. Binomial Equations

Example: Solve the equation $3x^7 = 5$

Theory: Bring the equation to the form $x^n = a$ and write the number a as a complex number, of the form $r(cos(\alpha) + i \cdot sin(\alpha))$. Use if necessary the forms $1 = cos(0) + i \cdot sin(0)$, respectively $-1 = cos(\pi) + i \cdot sin(\pi)$. We obtain $x^n = r(cos(\alpha) + i \cdot sin(\alpha))$ and apply the formula for the radical of a complex number and obtain the roots:

$$x_k = \sqrt[n]{r}(cos\frac{\alpha + 2k\pi}{n} + i \cdot sin\frac{\alpha + 2k\pi}{n}), k = 0, 1 \cdots n - 1$$

For example for $3x^7 = 5$ we write $x^7 = \frac{5}{3} \cdot 1$, that is $x^7 = \frac{5}{3} \cdot (cos(0) + i \cdot sin(0))$ therefore

$$x_k = \sqrt[7]{\frac{5}{3}}(cos\frac{0 + 2k\pi}{7} + i \cdot sin\frac{0 + 2k\pi}{7}), k = 0, 1 \cdots 6$$

Type 10. Other Methods for Solving Higher Degree Equations

- Using Bézout's theorem, for multiple roots (10th grade material)
- Using Viète's relations (10th grade material)
- Using Rolle's sequence (11th grade material, involves derivatives)
- Graphical solution (11th grade material, sometimes involves derivatives)

1. Calculation of Determinants

a) Determinants of order 1:

$$\begin{vmatrix} a \end{vmatrix} = a$$

b) Determinants of order 2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot d - b \cdot c$$

c) Determinants of order 3:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot e \cdot i + d \cdot h \cdot c + b \cdot f \cdot g - c \cdot e \cdot g - b \cdot d \cdot i - f \cdot h \cdot a$$

This method is called *triangle rule*

d) Determinants of order >= 4:

There is no rule like the previous rules, but we proceed as follows:

Let for example the following determinant of order 4:

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix}$$

We **expand** the determinant along a row or along a column. It should be noted that any row can be chosen, respectively any column, the result obtained is the same. That is, if we expand along row 1 it is ok, or if along row 2 it is also ok, if we expand along column 1 it is ok, along column 2 it is also ok, etc.

We choose for example to expand along row 1. The expansion of the determinant along row 1 is the following:

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} =$$

$$(-1)^{1+1} \cdot a \cdot \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} + (-1)^{1+2} \cdot b \cdot \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} +$$

$$(-1)^{1+3} \cdot c \cdot \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} + (-1)^{1+4} \cdot d \cdot \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

It is observed that starting from a determinant of order four, we arrived at determinants of order three, which we can calculate.

Practically we managed to reduce the degree of the determinant by one unit.

In general, to calculate a determinant of order n, by this method we arrive at determinants of order *n-1*, then applying the method again, determinants of order *n-1* are reduced to determinants of order *n-2*, etc. until we reach determinants of order 3, for which we have an effective calculation method.

This "reduced" determinant, for example $\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix}$ for the element located at the intersection of row 1 with column 1, is called **minor** and is denoted by Γ_{11}. In general at the intersection of row i with column j, we find the element a_{ij}, having the minor Γ_{ij}.

The expansion of a determinant along a certain row or along a certain column, represents in fact a sum of groupings of the form $(-1)^{i+j} \cdot a_{ij} \cdot \Gamma_{ij}$ for that row, respectively column.

The expansion of a determinant can be expressed briefly as follows:

$$d = \sum (-1)^{i+j} \cdot a_{ij} \cdot \Gamma_{ij}$$

2. Properties of Determinants

It is observed that a determinant is easier to calculate if the row (or column) along which the expansion is chosen has as many elements equal to zero as possible.

For this reason, it is preferable that for the calculation of a determinant we choose a row (or column), make as many zeros as possible on that row (or column) and only then expand the determinant.

To make zeros on a row (or column) the following properties can be used:

a) If a row is added to another row in a determinant, the value of the determinant is the same.

If we denote for example row i with L_i and row j with L_j, it means that the operation $L_i = L_i + L_j$ is correct

b) If a row is subtracted from another row in a determinant, the value of the determinant is the same, that is the operation $L_i = L_i - L_j$ is correct

c) If in a determinant a row is multiplied by any real number, for example α and then the obtained value is added to another row, the value of the determinant is the same, that is the operation $L_i = L_i + \alpha \cdot L_j$ is correct.

It is observed that for $\alpha = +1$ case **a)** is obtained and for $\alpha = -1$ case **b)** is obtained.

It should be noted that α can be any real number, for example $\alpha = \frac{1}{3}$ is ok.

In conclusion, remember that the following relations are correct:

$$a)\ \ L_i = L_i + L_j \quad b)\ \ L_i = L_i - L_j \quad c)\ \ L_i = L_i + \alpha \cdot L_j$$

Similarly for columns, that is denoting column with C, the following operations are correct:

$$a)\ \ C_i = C_i + C_j \quad b)\ \ C_i = C_i - C_j \quad c)\ \ C_i = C_i + \alpha \cdot C_j$$

3. From the above the following consequences result:

1) A determinant has value zero if it has on a row (or on a column) all elements equal to zero.

2) A determinant has value zero if it has two identical rows (or two identical columns).

3) A determinant has value zero if it has two proportional rows (or two proportional columns).

4) A common factor can be extracted from a row (or column) of a determinant, that is for example, the following operation is correct:

$$\begin{vmatrix} \alpha \cdot a & b \\ \alpha \cdot c & d \end{vmatrix} = \alpha \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

respectively the following operation:

$$\begin{vmatrix} a & \alpha \cdot b \\ c & \alpha \cdot d \end{vmatrix} = \alpha \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

SHEET 20: Matrices

1. The Notion of Matrix

Definition: A matrix represents a table of elements.

Let for example the matrix X=$\begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix}$ which has 2 rows and three columns, so we can say that X belongs to the set of matrices with 2 rows and 3 columns, that is X $\in \mathcal{M}_{2,3}$ It should be mentioned that the first index represents the number of rows and the second index represents the number of columns. In general, the set of matrices with m rows and n columns is denoted by $\mathcal{M}_{m,n}$

- If m=n, that is the number of rows equals the number of columns, the matrix is called *square matrix*, and the set of matrices is not written $\mathcal{M}_{m,m}$ but more simply, $\mathcal{M}_m$. If m is different from n, the matrix is also called rectangular matrix.

- If m=1, that is the matrix has a single row, the matrix is called *row matrix* and can be written for example X $\in \mathcal{M}_{1,n}$.

- If n=1, that is the matrix has a single column, the matrix is called *column matrix* and can be written for example X $\in \mathcal{M}_{m,1}$.

- It is observed that the row matrix and the column matrix represent in fact vectors.

2. Operations with Matrices

Let the matrices:
A=$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and B=$\begin{vmatrix} A & B \\ C & D \end{vmatrix}$

- **Matrix Addition:** A+B=$\begin{vmatrix} a+A & b+B \\ c+C & d+D \end{vmatrix}$
- **Matrix Subtraction:** A - B=$\begin{vmatrix} a-A & b-B \\ c-C & d-D \end{vmatrix}$
- **Multiplication of Matrix by Scalar:** $\alpha\cdot$ A=$\begin{vmatrix} \alpha\cdot a & \alpha\cdot b \\ \alpha\cdot c & \alpha\cdot d \end{vmatrix}$, for α scalar.
- **Multiplication of Two Matrices:**

Let the matrices: A=$\begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix}$ and B=$\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & I \end{vmatrix}$

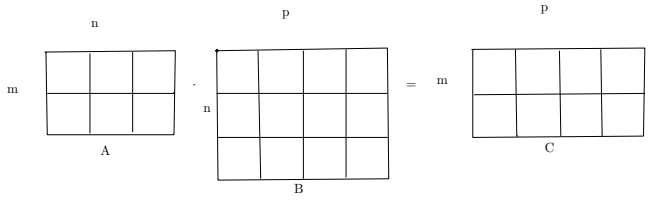
$$\text{A}\cdot\text{B}=\begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix}\cdot\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & I \end{vmatrix}=$$

$$=\begin{vmatrix} aA+bD+cG & aB+bE+cH & aC+bF+cI \\ dA+eD+fG & dB+eE+fH & dC+eF+fI \end{vmatrix}$$

It can be remembered more easily by looking at it this way: A·B=
$\begin{vmatrix} L_1C_1 & L_1C_2 & L_1C_3 \\ L_2C_1 & L_2C_2 & L_2C_3 \end{vmatrix}$, where L_i=row i, and C_j=column j

Analyzing the algorithm for multiplying two matrices, it is observed that if a matrix with *m rows* and *n columns* is multiplied by another matrix having *n rows* and *p columns*, the resulting matrix will have *m rows* and *p columns*. For the multiplication to be performed, it is necessary that the number of rows of the second matrix be equal to the number of columns of the first matrix.

For example for the multiplication A · B = C (where A has m rows and n columns, and B has n rows and p columns) the following rules presented in the following diagram are respected:



It is observed that matrix multiplication is NOT COMMUTATIVE

• **Division of Two Matrices:**

There is no division of two matrices. To "simulate matrix division" the notion of *inverse matrix* is used, denoted by A^{-1} (for the initial matrix A), which will be presented later.

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3. Special Matrices

Let for example matrices of 3 rows and 3 columns, that is matrices from $\mathcal{M}_3$

a) zero matrix. For $\mathcal{M}_3$:

$$O_3 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

In general, the zero matrix for $\mathcal{M}_n$, is denoted by O_n and is a square matrix, with n rows and n columns, having all elements zero.

The zero matrix has the property that it is neutral with respect to matrix addition, that is $A+O_n = O_n+A=A$ for any $A \in \mathcal{M}_n$.

b) identity matrix. For $\mathcal{M}_3$:

$$I_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

In general, the identity matrix for $\mathcal{M}_n$, is denoted by I_n and is a square matrix, with n rows and n columns, having all elements zero except the elements on the main diagonal which all have value 1.

The identity matrix has the property that it is neutral with respect to matrix multiplication, that is $A \cdot I_n = I_n \cdot A = A$ for any $A \in \mathcal{M}_n$.

---

1. Inverse Matrix

Let for example the square matrix of order 3, $A=\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

To determine the inverse matrix, denoted by A^{-1}, we proceed as follows:

1. Calculate the determinant of matrix A.

a) If det(A)=0, it means that the inverse matrix A^{-1} does not exist, or in other words, matrix A is not invertible. In this situation, the process stops here.

b) If det(A) different from zero, it means that the inverse matrix A^{-1} exists, or in other words, matrix A is invertible. Continue with the next step.

2. Form the transpose matrix, denoted by A^t, namely row 1 from A becomes column 1 in A^t, row 2 from A becomes column 2 in A^t, etc. Thus we obtain:

$$A^t = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

3. Form the adjoint (or reciprocal) matrix, denoted by A^*, **from $\mathbf{A^t}$**, which will have as many elements as A^t, and will have the following form:

$$A^* = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}$$

where $x_{11}=(-1)^{(1+1)} \cdot \begin{vmatrix} e & h \\ f & i \end{vmatrix} = 1 \cdot (e \cdot i - h \cdot f) = ei - hf =$ number.

$x_{12}=(-1)^{(1+2)} \cdot \begin{vmatrix} b & h \\ c & i \end{vmatrix} = -1 \cdot (b \cdot i - h \cdot c) = -bi + hc =$ number, etc.

Warning: Do not confuse with the method of expanding a determinant, where the generic factor $(-1)^{i+j} \cdot a_{ij} \cdot \Gamma_{ij}$ was used and here a_{ij} is not used

4. Form A^{-1} with the formula:

$$A^{-1}=\frac{A^*}{det(A)} = \frac{1}{det(A)} \cdot \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = \begin{vmatrix} \frac{x_{11}}{det(A)} & \frac{x_{12}}{det(A)} & \frac{x_{13}}{det(A)} \\ \frac{x_{21}}{det(A)} & \frac{x_{22}}{det(A)} & \frac{x_{23}}{det(A)} \\ \frac{x_{31}}{det(A)} & \frac{x_{32}}{det(A)} & \frac{x_{33}}{det(A)} \end{vmatrix}$$

The reason is observed why at point 1 it was concluded that A^{-1} does not exist if det(A)=0 was obtained.

5. Optional step, but recommended if there is time, namely verification of A^{-1}. Perform the multiplication $A \cdot A^{-1}$ and verify if indeed the identity matrix is obtained, in the case of the presented example $I_3=\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ For the general case, if I_n is obtained for a square matrix of order n, it means that A^{-1} was calculated correctly, otherwise it means there was an error and the calculation must be verified.

Notes

1. It is observed that if it is required to verify whether a matrix is invertible (or in other words if an inverse matrix exists), the problem is reduced to verifying the determinant of the matrix. If the determinant of the matrix is different from zero it means that the matrix is invertible and if the determinant of the matrix equals zero it means that the matrix is not invertible.

2. Problems that use the inverse matrix are generally of the following types:

- A matrix is given and it is required to determine the inverse matrix.

- A matrix is given and it is required to verify whether the matrix is invertible.

- Solving matrix equations.

- Solving systems of linear equations by the matrix method.

---

2. Rank of a Matrix

Let for example the following rectangular matrix:

$$X = \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{vmatrix}$$

By definition, the rank of a matrix represents the order of the largest square determinant, different from zero, extracted from the matrix.

In our case, the matrix being of type 3x4, the maximum order we can *hope for* is 3x3. We extract from the matrix a determinant of order 3x3, calculate it and verify if it is different from zero. If it is different from zero, then the rank of the matrix is 3 and we stop. If the determinant equals zero, we choose another determinant of order 3x3 and proceed similarly.

If all determinants of order 3x3 equal zero, *we lower our expectations* and try with determinants of order 2x2 in a similar way.

If all determinants of order 2x2 equal zero, *we lower our expectations* and try with determinants of order 1x1.

It is observed that in *the worst case* we will find *at least* one determinant different from zero of order 1, except for the situation where the matrix has all elements zero, which would mean that the rank equals zero. From the above it can be sensed that the algorithm is finite.

Notes

1. It is observed that the presented method starts from a *top-down* approach to the matrix, in the sense that it begins with the square determinants extracted from the matrix, of maximum rank.

2. Care must be taken to consider all determinants of a certain dimension, that is *not to miss* any determinants.

3. Problems that use the rank of a matrix are generally of the following types:

- A matrix is given and it is required to determine its rank.

- A matrix is given and it is required to discuss the rank as a function of a parameter.

- Solving systems of linear equations.

---

1. Systems of Equations Solved by Matrix Method

Let for example the system: $\begin{cases} x+y+z & =7 \\ 3x+7y+5z & =2 \\ 6x-3y+z & =1 \end{cases}$

Matrix solution of the system is done following these steps:

- Form the coefficient matrix of the system, in our case A= $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 7 & 5 \\ 6 & -3 & 1 \end{bmatrix}$

- Form the column matrix with the unknowns of the system, X= $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- Form the column matrix with the constant terms of the system, B= $\begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}$

- The system is equivalent to the following matrix equation:

$$\mathrm{A}\cdot \mathrm{X} = \mathrm{B}$$

- Solve the previous matrix equation in the usual way, that is calculate A^{-1}, multiply the matrix equation on the left by A^{-1} and obtain $\mathrm{X}=\mathrm{A}^{-1}\cdot \mathrm{B}$. We obtain the column matrix X, from which we extract x,y,z.

Note This method is used rarely, possibly only if it is explicitly requested in the statement to solve by this method.

2. Cramer's Rule

Let for example the system: $\begin{cases} x+y+z & =7 \\ 3x+7y+5z & =2 \\ 6x-3y+z & =1 \end{cases}$

The following steps are performed:

- Form the coefficient matrix of the system, in our case A= $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 7 & 5 \\ 6 & -3 & 1 \end{bmatrix}$

- Calculate the determinant of matrix A, denote it for example by Δ (delta).

• If Δ =0, it means that the system cannot be solved by Cramer's rule and the process stops here.

• If $\Delta \neq 0$, it means that the system can be solved by Cramer's rule and we continue with the next stage.

- Calculate Δ_x, by replacing the column with the coefficients of x from the system matrix, with the column of constant terms, that is we obtain: $\Delta_x = \begin{bmatrix} \mathbf{7} & 1 & 1 \\ \mathbf{2} & 7 & 5 \\ \mathbf{1} & -3 & 1 \end{bmatrix}$

- Calculate Δ_y and Δ_z similarly.

- Calculate x=$\frac{\Delta_x}{\Delta}$ y=$\frac{\Delta_y}{\Delta}$ z=$\frac{\Delta_z}{\Delta}$ the solution of the system being (x,y,z).

The reason is observed why a system having $\Delta = 0$ cannot be solved by Cramer's rule, namely because Δ appears in the denominator.

3. Study of System Compatibility

Let for example the system: $\begin{cases} 3x+y+2z &= 8 \\ 2x-3y+z &= 1 \\ 5x+9y+4z &= 22 \end{cases}$

The following steps are performed:

- Form the coefficient matrix of the system, in our case A= $\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & 1 \\ 5 & 9 & 4 \end{bmatrix}$

- Calculate the determinant of matrix A, denote it for example by Δ (delta).

• If $\Delta \neq 0$, it means that the system is **compatible determinate**, it can be solved by Cramer's rule and the present method is not necessary. Solve by Cramer's rule presented earlier and the process stops here.

• If $\Delta = 0$, it means that the system cannot be solved by Cramer's rule and we perform the following steps. In our case from calculation we obtain $\Delta = 0$.

- Extract from the system matrix **a square determinant of the largest order, different from zero**, called **principal determinant**, which we denote for example by δ (small delta).

For example $\delta = \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix}$ = -9-2=-11 $\neq 0$

- Establish the **principal equations**, in this case equation 1 and equation 2 and the **secondary equations**, in this case equation 3. Establish the **principal unknowns**, in this case x and y and the **secondary unknowns**, in this case z.

- Form the **characteristic determinant**, denoted by Δ_c, *by bordering the principal determinant δ with a row formed from the corresponding coefficients from* **one of** the secondary equations and with *a column formed from the corresponding constant terms.* For our case, we obtain $\Delta_c = \begin{bmatrix} \mathit{3} & \mathit{1} & \mathbf{8} \\ \mathit{2} & \mathit{-3} & \mathbf{1} \\ \mathbf{5} & \mathbf{9} & \mathbf{22} \end{bmatrix}$

- Calculate Δ_c, (in our case $\Delta_c = 0$) and interpret as follows:

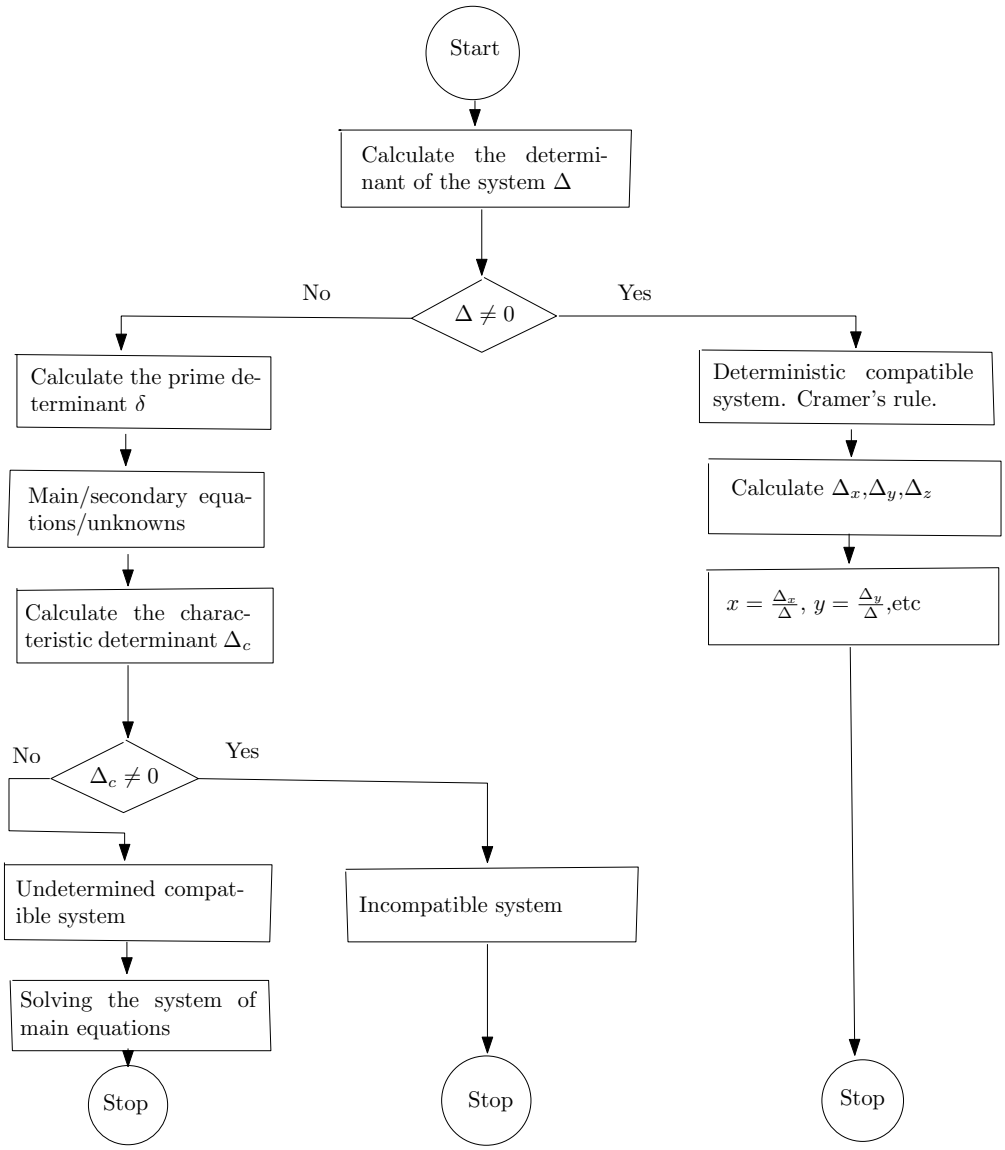
• If $\Delta_c \neq 0$ we conclude that the system is **incompatible** and the process stops here.

• If $\Delta_c = 0$ we conclude that the system is **compatible indeterminate** (in this case, **simply indeterminate compatible system** because we have a single secondary unknown) and we continue with the next step:

- Solve **the system formed from the principal equations namely extract the principal unknowns as functions of the secondary unknowns**. Practically write the system formed from the secondary equations, move the secondary unknowns to the member with the constant terms and extract the principal unknowns as functions of the secondary unknowns. In our case, solve the system $\begin{cases} 3x+y &= 8-2z \\ 2x-3y &= 1-z \end{cases}$ by usual methods (reduction method or substitution or even Cramer) and extract x=f(z) and y=f(z), z $\in$ R. Specifically we obtain x=$\frac{25-7z}{11}$, y=$\frac{13-z}{11}$, z $\in$ R.

SHEET 23: Compatibility of Systems of Equations = Graphically

Graphical representation of the stages for analyzing the compatibility of a system with n equations and n unknowns:



It is recommended to analyze systems following this model, possibly with adaptation required by the specifics of the problem.

Note:

The graphical representation presented earlier uses the rules for representing **logical flowcharts**, frequently used in computer science.

Briefly, the main conventions for representing logical flowcharts:

The diamond represents a decision structure, having two exits, one for true condition and another for false condition.
The rectangle frames an action.
The circle is used to frame the beginning (*Start)* of the algorithm, respectively the end (*Stop)*.

This graphical representation is very suggestive for people familiar with reading a logical flowchart.

People who are not accustomed to this mode of representation - in case they encounter difficulties - can ignore this graphic and study *Sheet 22 and Sheet 24*, the result they will arrive at in the end being the same.

SHEET 24: Particular Systems of Equations

1. Discussing the Nature of a System According to Real Parameters

Follow the stages presented in the compatibility study graph and treat by cases, according to the values of the parameters. For example if we have a system of 3 equations with 3 unknowns, with a parameter m. We calculate the determinant of the system and obtain for example $\Delta = (m-5)\cdot(m+7)$. We proceed as follows:

Case 1 For $(m-5)\cdot(m+7) \neq 0$, that is for m$\neq$ 5 and m$\neq -7$ the system is compatible determinate and Cramer's rule is applied. **Extract effectively as functions of m** Δ then $\Delta_x, \Delta_y, \Delta_z$ etc. then x,y,z.

Case 2 For m=5 **substitute effectively m=5** in the initial system, obtain a system without parameters, which is solved absolutely normally like any ordinary system without parameters. Of course $\Delta = 0$ will be obtained and the corresponding path will be followed.

Case 3 For m=-7 **substitute effectively m=-7** in the initial system, obtain a system without parameters, which is solved absolutely normally like any ordinary system without parameters. Of course $\Delta = 0$ will be obtained and the corresponding path will be followed.

If Δ_c is obtained as a function of the parameter, (for example $\Delta_c = m-9$) we will separate again by cases, that is:

Case a) For $m - 9 \neq 0$ that is $m \neq 9$, the system is incompatible and stop.

Case b) For $m - 9 = 0$ that is $m = 9$, the system is compatible and indeterminate and we continue according to the compatibility graph, by solving the system formed from the principal equations.

Example Discuss the nature of the system according to the values of parameter m and solve the system:

$$\begin{cases} x - my + z & = 1 \\ x - y + z & = 1 \\ mx + m^2y - z & = m^2 \end{cases}$$

2. Homogeneous Systems

A homogeneous system is by definition a system that has all constant terms equal to zero.

All properties and known working method from systems are preserved, the only novelty being that a homogeneous system admits always the solution x=y=z= $\cdots$ =0 called **trivial solution** or **null solution**.

Analyzing the compatibility study graph, it is observed that if $\Delta \neq 0$ it results that the system is compatible determinate, that is it admits a **unique** solution. Since the homogeneous system always admits the trivial solution, it means that for $\Delta \neq 0$, the trivial solution is the **unique solution**. Therefore if for a homogeneous system it is required to impose the condition that the system **does not contain only the trivial solution**, the condition $\Delta = 0$ must be imposed.

Example Solve the following homogeneous system:

$$\begin{cases} x+2y+z+t &= 0 \\ 2x+y+z+2t &= 0 \\ x+2y+2z+t &= 0 \\ x+y+z+t &= 0 \end{cases}$$

---

3. Systems with Number of Equations Different from Number of Unknowns

One of the following two situations may appear:

a) The number of equations is less than the number of unknowns.
Example Solve the following system:

$$\begin{cases} x-2y+z+t &= 1 \\ x-2y+z-t &= -1 \\ x-2y+z+5t &= 6 \end{cases}$$

b) The number of equations is greater than the number of unknowns.
Example Solve the following system:

$$\begin{cases} x+2y &= 1 \\ 6x-8y &= 1 \\ 5x+2y &= 3 \end{cases}$$

It is recommended as an exercise, analyzing the compatibility study graph for each of these situations.

As general recommendations:

• For case a) the system even if it is compatible cannot also be determinate, because we are in the situation where we have fewer equations than unknowns.

• For case b) we have more equations than unknowns, so the system could be determinate, but the obtained solutions must be verified for all equations to be sure that the system is compatible.

Theory

The following are considered evident:

a) $\lim\limits_{n\to\infty} n = \infty$. In general, $\lim\limits_{n\to\infty} n^k = \infty$ for k > 0.

b) $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. In general $\lim\limits_{n\to\infty} \frac{1}{n^k} = 0$ for $k > 0$.

c) $\lim\limits_{n\to\infty} (\frac{2}{3})^n = 0$. In general, $\lim\limits_{n\to\infty} a^n = 0$ for a = subunitary.

Types of Sequence Limits

Type 1 = Simple Fractions

Examples:

a) $\lim\limits_{n\to\infty} \frac{2n^4+3n^2+5n+1}{4n^3+2n^2+3n+3}$ b) $\lim\limits_{n\to\infty} \frac{2n^4+3n^2+5n+1}{4n^6+2n^2+3n+3}$ c) $\lim\limits_{n\to\infty} \frac{2n^4+3n^2+5n+1}{4n^4+2n^2+3n+3}$

Idea: Factor out, both in the denominator and in the numerator, **n** to the highest power.

Type 2 = Consecrated Sums

Examples:

a) $\lim\limits_{n\to\infty} \frac{1+2+3+\cdots n}{3n^2+2n+1}$ b) $\lim\limits_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{3n^2+2n+1}$ c) $\lim\limits_{n\to\infty} \frac{1\cdot2+2\cdot3+\cdots n\cdot(n+1)}{4n^5+5}$

Idea: Reduce the sum according to known rules, after which the problem usually becomes of type 1.

Type 3 = Radicals

Examples:

a) $\lim\limits_{n\to\infty} (\sqrt{n+1} - \sqrt{n+5})$ b) $\lim\limits_{n\to\infty} (\sqrt{n+1} - 3n)$ c) $\lim\limits_{n\to\infty} (\sqrt[3]{n+1} - \sqrt[3]{n+2})$

Idea: Multiply the grouping $(\sqrt{a} - \sqrt{b})$ by $(\sqrt{a} + \sqrt{b})$ to eliminate the radical from the numerator, where a-b will be obtained, the problem usually becoming of type 1. Similarly, for the grouping $(\sqrt{a} + \sqrt{b})$, multiply by $(\sqrt{a} - \sqrt{b})$.

Type 4 = Powers

Examples:

a) $\lim\limits_{n\to\infty} \frac{4\cdot5^n+6\cdot3^n}{5^n+3\cdot2^n}$ b) $\lim\limits_{n\to\infty} \frac{4\cdot7^{n+3}+6\cdot3^{n+1}}{7^{n+2}+4\cdot3^{n+4}}$ c) $\lim\limits_{n\to\infty} \frac{4\cdot5^n+6\cdot6^n}{7^n+3\cdot8^n}$

Idea: Factor out, both in the denominator and in the numerator, the power a^n to the highest power, for example for example a), in the numerator factor out 5^n and in the denominator also 5^n.

Type 5 = Grouping type $(1+\frac{1}{n})^n$. Indeterminate 1^∞

Examples:

a) $\lim\limits_{n\to\infty}(1+\frac{3}{n-2})^{n+1}$ b) $\lim\limits_{n\to\infty}(\frac{n+1}{n-5})^{n+2}$ c) $\lim\limits_{n\to\infty}(\frac{n^2+3}{n^2-n})^{2n+1}$

Idea: Try to arrive at the formula $\lim\limits_{n\to\infty}(1+\frac{1}{n})^n = e$, respectively for the general case, $\lim\limits_{n\to\infty}(1+A)^{\frac{1}{A}} = e$, where A $\to$0 for n$\to\infty$.

Type 6 = Squeeze Theorem

Examples:

a) $\lim\limits_{n\to\infty}\frac{2^n}{n!}$ b) $\lim\limits_{n\to\infty}\frac{1}{n^2}\cdot sin(n+7)$ c) $\lim\limits_{n\to\infty}\frac{1}{\sqrt{1+n^2}}+\frac{1}{\sqrt{2+n^2}}+\cdots+\frac{1}{\sqrt{n+n^2}}$

Idea: Use the **squeeze theorem**:

Let a sequence a_n and a limit l. If $b_n <= a_n <= c_n$ and both the sequence $b_n \to l$ and $c_n \to l$, then $a_n \to l$. If l=0, the rule is also called **majorization criterion**.

Theory

The following are considered known:

a) Method of calculating limits of sequences

b) Decomposition $ax^2+bx+c = a(x-x_1)(x-x_2)$ and similarly for any polynomial of degree n.

c) $\lim\limits_{x\to 0} \frac{sin(x)}{x} = 1$

Limits of functions can be calculated by two major methods: classical, that is without using derivatives and using l'Hôpital's rule which involves using derivatives. Some problems are solved more easily by one method, other problems by the other. There is no recipe to indicate which is the most suitable method for a certain problem.

A) Types of Classical Function Limits

(without using derivatives)

Type 1 = Limits with $x \to \pm\infty$

Examples:

a) $\lim\limits_{x\to\infty} \frac{5x^4+4x^2+2x+1}{3x^2+2x+1}$ b) $\lim\limits_{x\to\infty} (\frac{x+3}{x-5})^{2x+1}$ c) $\lim\limits_{x\to-\infty} \frac{\sqrt{x^2+3}+3x}{2x+1}$

Idea: Work similarly to limits of sequences, where we had $n \to \pm\infty$.

Type 2 = Limits with $x \to x_0$ and $x-x_0$ appears in denominator

Examples:

a) $\lim\limits_{x\to 5} \frac{1}{x-5}$ b) $\lim\limits_{x\to 7} \frac{-3(x^2+2)}{x-7}$ c) $\lim\limits_{x\to 3} \frac{2}{5+\frac{-6}{x-3}}$

Idea: Make a table and finally obtain the limit to the left of x_0 and the limit to the right of x_0.

Type 3 = Limits from Fractions of Polynomials

Examples:

a) $\lim\limits_{x\to 7} \frac{x^2-49}{x-7}$ b) $\lim\limits_{x\to 2} \frac{x^3-8}{x-2}$ c) $\lim\limits_{x\to 3} \frac{x^4-81}{x-3}$

Idea: Try to decompose both the numerator and the denominator according to the formula of the form $ax^2+bx+c = a(x-x_1)(x-x_2)$ (corresponding to the degree of the polynomial), or possibly according to other formulas, after which try to simplify the factor $x-x_0$.

Type 4 = Limits of Trigonometric Functions

Examples:

a) $\lim\limits_{x\to 0} \frac{sin(7x)}{3x}$ b) $\lim\limits_{x\to 0} \frac{tg(3x)}{tg(7x)}$ c) $\lim\limits_{x\to 3} \frac{sin(x-3)}{4(x-3)}$

Idea: Use the formula $\lim\limits_{x\to 0} \frac{sin(x)}{x} = 1$ and the usual formulas from trigonometry. Keep in mind that sin(0)=0 and cos(0)=1.

The methods presented above are part of the category of **calculating function limits by classical method** in the sense that they **do not** use derivatives. Function limits can also be calculated using derivatives, using l'Hôpital's rule, presented below.

B) l'Hôpital's Theorem

Function limits using derivatives

Calculating function limits can **sometimes** be simplified by using l'Hôpital's theorem.

l'Hôpital's Theorem: For indeterminate forms of type $\frac{\infty}{\infty}$ or $\frac{0}{0}$ the following relation is true:

$$\lim_{x\to x_0}\frac{f}{g}=\lim_{x\to x_0}\frac{f'}{g'}$$

Particular Cases In the case where the indeterminate form is not of type $\frac{\infty}{\infty}$ or $\frac{0}{0}$, the given expression must be transformed into one of these forms. Thus we proceed if we have indeterminate forms of type:

1. $[0\cdot\infty]$
2. $[\infty-\infty]$
3. $[1^{\infty},\infty^{0},etc]$ in general indeterminate forms of the form f^{g}

Sometimes it is simpler to calculate a function limit by l'Hôpital, other times it is simpler by classical methods (classical meaning without derivatives, meaning without l'Hôpital). **There is no recipe** to recommend for a given expression whether it is easier with one method or another.

For indeterminate forms different from $\frac{\infty}{\infty}$ or $\frac{0}{0}$, we proceed as follows:

---

1. For indeterminate form $[0\cdot\infty]$

Let us calculate $\lim\limits_{x\to x_0} f\cdot g$, where $f\to 0$ and $g\to\infty$.

We **invert** one of the functions, for example:

$f\cdot g=\frac{g}{\frac{1}{f}}$ $[\frac{\infty}{\infty}]$ and now we can apply l'Hôpital's rule:

$$\lim_{x\to x_0} f\cdot g=\lim_{x\to x_0}\frac{g}{\frac{1}{f}}\ [\frac{\infty}{\infty}]\ =\lim_{x\to x_0}\frac{g^{'}}{(\frac{1}{f})'}$$

---

2. For indeterminate form $[\infty-\infty]$

Let us calculate $\lim\limits_{x\to x_0}(f-g)$, where $f\to\infty$ and $g\to\infty$.

We transform as follows:

$f-g=f\cdot g\frac{(f-g)}{f\cdot g}=f\cdot g\cdot(\frac{f}{fg}-\frac{g}{fg})=f\cdot g\cdot(\frac{1}{g}-\frac{1}{f})=[\infty\cdot 0]=\frac{\frac{1}{g}-\frac{1}{f}}{\frac{1}{f\cdot g}}=[\frac{0}{0}]$

and now we can apply l'Hôpital's rule:

$$\lim_{x\to x_0}(f-g)=\lim_{x\to x_0}\frac{\frac{1}{g}-\frac{1}{f}}{\frac{1}{f\cdot g}}=\lim_{x\to x_0}\frac{(\frac{1}{g}-\frac{1}{f})'}{(\frac{1}{f\cdot g})'}$$

---

3. For indeterminate forms of type f^{g}. For example 1^{∞}, ∞^{0}, etc.

We use the formula $A=e^{ln(A)}$, and take A=f^{g}.

SHEET 27: Applications of Function Limits

Continuity, Derivatives, Differentiability, Asymptotes

1. Continuous Functions

Any elementary function is continuous on an interval.
The study of continuity is posed at points, for example at point $x = x_0$. We proceed as follows:

1. - Calculate the left limit at x_0, $l_s|_{x=x_0} = lim_{x \to x_0 (x<x_0)} f(x)$
2. - Calculate the right limit at x_0, $l_d|_{x=x_0} = lim_{x \to x_0 (x>x_0)} f(x)$
3. - Calculate the value of the function at point x_0, that is $f(x_0)$
4. - If the left limit at x_0 equals the right limit at x_0 and equals $f(x_0)$, then the function f(x) is continuous at point x_0, otherwise f(x) is not continuous at point x_0.

Briefly, the condition for continuity at a point x_0 is that

$$l_s|_{x=x_0} = l_d|_{x=x_0} = f(x_0)$$

Since the study of continuity is done at points, continuity problems appear in problems where f(x) is given in the form of *braces*, the study being performed at points where the function changes its form.

2. Definition of Derivative

The derivative of function f(x) at point x_0 is denoted by $f(x)'_{x=x_0}$ and is defined as follows:

$$f(x)'_{x=x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

By applying the previous formula to known functions, the derivative table was obtained.

3. Differentiable Functions

Elementary functions are differentiable on intervals.
The study of differentiability is posed at points, for example at point $x = x_0$. We proceed as follows:

The function f(x) is differentiable at point $x = x_0$, if the derivative to the left at x_0 and the derivative to the right at x_0 are **equal and finite**.

a) Left derivative at point x_0 is denoted by $f's_{(x=x_0)}$ and is calculated with the definition formula of derivative:

$$f(x)'_s(x = x_0) = \lim_{x \to x_0, x<x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

It can be calculated with l'Hôpital's rule because there is an indeterminate form $\frac{0}{0}$ and we obtain

$$f(x)'_s(x = x_0) = \lim_{x \to x_0, x<x_0} f'(x)$$

b) Right derivative at point x_0 is denoted by $f^{'}d_{(x=x_0)}$ and is calculated with the definition formula of derivative:

$$f(x)_d^{'}(x=x_0) = \lim_{x\to x_0, x>x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

It can be calculated with l'Hôpital's rule because there is an indeterminate form $\frac{0}{0}$ and we obtain

$$f(x)_d^{'}(x=x_0) = \lim_{x\to x_0, x>x_0} f^{'}(x)$$

In conclusion, calculate the left derivative at point x_0 and the right derivative at x_0. If these two are **equal and finite**, then the function f(x) is differentiable at point x_0, otherwise f(x) is not differentiable at point x_0.

Since the study of differentiability is done at points, differentiability problems appear in problems where f(x) is given in the form of *braces*, the study being performed at points where the function changes its form.

---

4. Asymptotes

Three types of asymptotes may exist:
• **Horizontal Asymptote** A *horizontal* line y=a is called horizontal asymptote at $+\infty$ if $lim_{x\to+\infty}$ f(x) = a

Similarly, a *horizontal* line y=b is called horizontal asymptote at -∞ if $lim_{x\to-\infty}$ f(x) = b
If $+\infty$ is not part of the domain of definition, then we surely do not have a horizontal asymptote at $+\infty$. Similarly for -∞.
• **Vertical Asymptote** It is a *vertical* line x=c, if $lim_{x\to c}$ f(x) = +-∞
• **Oblique Asymptote** Study at $+\infty$:

If $+\infty$ is not part of the domain of definition, then we surely do not have an oblique asymptote at $+\infty$. If $+\infty$ is part of the domain of definition but we have already established that we have a horizontal asymptote at $+\infty$ then we surely do not have an oblique asymptote at $+\infty$, because the function cannot have simultaneously two asymptotes at $+\infty$. If $+\infty$ belongs to the domain of definition and $lim_{x\to+\infty}$ f(x) = +-∞ it means that at $+\infty$ we do not have a horizontal asymptote. In this case we verify *whether there possibly exists* an oblique asymptote at $+\infty$. We calculate $m = lim_{x\to+\infty}\frac{f(x)}{x}$ If we obtain *m finite*, we continue with calculating $n = lim_{x\to+\infty}(f(x) - m\cdot x)$ If n is also finite, we can conclude that there exists an oblique asymptote at $+\infty$, namely the *oblique* line $y = m\cdot x + n$.

We study similarly the existence of oblique asymptote at -∞.

Practically we proceed as follows:
1. Establish the domain of definition, written as union of intervals.
2. Calculate the values of the function, respectively the limits of the function *at the endpoints of the domain of definition.*
3. Establish asymptotes by analyzing the values of the function, respectively the values of the obtained limits and the definitions of asymptotes, presented earlier. It is observed that horizontal and oblique asymptotes can appear at + ∞ or at - ∞, and vertical asymptotes can appear at interval endpoints different from +- ∞. Some functions have asymptotes, others do not.

SHEET 28: Derivatives

1. Definition of Derivative

The derivative of function f(x) at point x_0 is denoted by $f(x)'_{x=x_0}$ and is defined as follows:

$$f(x)'_{x=x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

2. Derivative Table

Function	Derivative
k(=constant)	0
x	1
x^n	$n \cdot x^{n-1}$
$\sqrt{x}$	$\frac{1}{2 \cdot \sqrt{x}}$
$ln(x)$	$\frac{1}{x}$
a^x	$a^x \cdot ln(a)$
e^x	e^x
$sin(x)$	$cos(x)$
$cos(x)$	$-sin(x)$
$tg(x)$	$\frac{1}{cos(x)^2}$
$ctg(x)$	$-\frac{1}{sin(x)^2}$
$arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$arctg(x)$	$\frac{1}{1+x^2}$
$arcctg(x)$	$-\frac{1}{1+x^2}$

3. Operations with Derivatives

$(f+g)^{'} = f^{'} + g^{'}$

$(f-g)^{'} = f^{'} - g^{'}$

$(f \cdot g)^{'} = f^{'} \cdot g + f \cdot g^{'}$

$(k \cdot f)^{'} = k \cdot f^{'}$, where k=constant

$(\frac{f}{g})^{'} = \frac{f^{'} \cdot g - f \cdot g^{'}}{g^2}$

• Advanced $(f^g)^{'} = f^g \cdot (g^{'} \cdot ln(f) + g \cdot \frac{f^{'}}{f})$

• Differentiation of composite functions: $f(u(v(x)))^{'} = f^{'}(u(v(x))) \cdot u^{'}(v(x)) \cdot v^{'}(x)$

4. Differentiable Functions

The function f(x) is differentiable at point $x = x_0$, if the derivative to the left at x_0 and the derivative to the right at x_0 are **equal and finite**.

a) Left derivative at point x_0 is denoted by $f^{'}s_{(x=x_0)}$ and is calculated with the definition formula of derivative:

$$f(x)^{'}_{s}(x = x_0) = \lim_{x \to x_0, x < x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

It can be calculated with l'Hôpital's rule because there is an indeterminate form $\frac{0}{0}$ and we obtain

$$f(x)^{'}_{s}(x = x_0) = \lim_{x \to x_0, x < x_0} f^{'}(x)$$

b) Right derivative at point x_0 is denoted by $f^{'}d_{(x=x_0)}$ and is calculated with the definition formula of derivative:

$$f(x)^{'}_{d}(x = x_0) = \lim_{x \to x_0, x > x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

It can be calculated with l'Hôpital's rule because there is an indeterminate form $\frac{0}{0}$ and we obtain

$$f(x)^{'}_{d}(x = x_0) = \lim_{x \to x_0, x > x_0} f^{'}(x)$$

In conclusion, calculate the left derivative at point x_0 and the right derivative at x_0. If these two are **equal and finite**, then the function f(x) is differentiable at point x_0, otherwise f(x) is not differentiable at point x_0.

5. Main Applications of Derivatives

• l'Hôpital's Theorem

• Lagrange's Theorem

• Rolle's Theorem, Cauchy's Theorem

• Concave, convex functions

• Graphical representation of functions

SHEET 29: Applications of Derivatives

Theorems of l'Hôpital, Rolle, Lagrange, Cauchy
Monotonicity, Concavity/Convexity, Function Graphs

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1. l'Hôpital's Theorem

Calculating function limits can **sometimes** be simplified by using l'Hôpital's theorem.

l'Hôpital's Theorem: For indeterminate forms of type $\frac{\infty}{\infty}$ or $\frac{0}{0}$ the following relation is true:

$$\lim_{x \to x_0} \frac{f}{g} = \lim_{x \to x_0} \frac{f'}{g'}$$

Particular Cases In the case where the indeterminate form is not of type $\frac{\infty}{\infty}$ or $\frac{0}{0}$, the given expression must be transformed into one of these forms. Thus we proceed if we have indeterminate forms of type:

1. $[0 \cdot \infty]$
2. $[\infty - \infty]$
3. $[1^{\infty}, \infty^0, etc]$ in general indeterminate forms of the form f^g

Sometimes it is simpler to calculate a function limit by l'Hôpital, other times it is simpler by classical methods (classical meaning without derivatives, meaning without l'Hôpital). There is no recipe to recommend for a given expression whether it is easier with one method or another.

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2. Rolle's Theorem

Let f(x) be a function, I an interval and two numbers a,b, where a < b and a,b ∈ I.

If:

1. The function f(x) is continuous for x ∈ [a,b]
2. The function f(x) is differentiable for x ∈ (a,b)
3. f(a)=f(b)

then there exists at least one point **c** between a and b, such that **f'(c)=0**.

---

3. Lagrange's Theorem

Let f(x) be a function, I an interval and two numbers a,b, where a < b and a,b ∈ I.

If:

1. The function f(x) is continuous for any x ∈ [a,b]
2. The function f(x) is differentiable for any x ∈ (a,b)

then there exists a point **c** between a and b, such that:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Consequences of Lagrange's theorem:

1. If f'(x) > 0 on an interval, then f(x) is **increasing** on that interval
2. If f'(x) < 0 on an interval, then f(x) is **decreasing** on that interval
3. If f'(x) $= 0$ on an interval, then f(x) is **constant** on that interval

Notes:

a) Extreme points: By studying the sign of f'(x) we can analyze the monotonicity of function f(x) (increasing/decreasing) and based on monotonicity we can establish the extreme points of the function (minimum respectively maximum).

b) Advanced: Geometric interpretation of Lagrange's theorem.

---

4. Cauchy's Theorem

Let f(x) and g(x) be functions, I an interval and two numbers a,b, with a<b and a,b $\in$ I.

If:

1. The functions f(x) and g(x) are continuous for x $\in$ [a,b]
2. The functions f(x) and g(x) are differentiable for x $\in$ (a,b)
3. The function g'(x) $\neq 0$ for x $\in$ (a,b)

then there exists a point **c** between a and b, such that:

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

It is observed that for the particular case g(x)=x, we obtain Lagrange's theorem.

---

5. Concavity, Convexity

New notions:

- Convex function, concave function.

- Inflection point, turning point.

Theory:

1. If f"(x) > 0, then f(x)= **convex** on that interval.
2. If f"(x) < 0, then f(x)= **concave** on that interval.

---

6. Graphical Representation of Functions

1. Determination of domain of definition, written as unions of intervals
2. Calculation of function values (function limits) at the endpoints of the domain of definition
3. Determination of asymptotes
4. Intersection with Ox axis (make y=0 and solve for x)
5. Intersection with Oy axis (make x=0 and solve for y)
6. First derivative: f'=0, solve equation, table, monotonicity, extreme points
7. Second derivative: f"=0, solve equation, table, concavity/convexity
8. General table
9. Draw graph

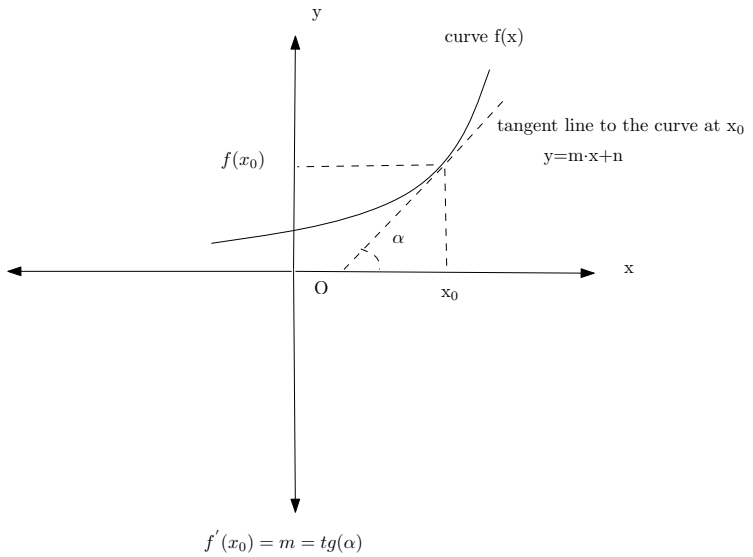
SHEET 30: Geometric Interpretation of the Derivative

Geometric Interpretation of the Derivative:

• Recall that the derivative of a function f(x) at a point x_0 is by definition:

$$f^{'}(x)_{x=x_0} = f^{'}(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

• The derivative of a function f(x) at point x_0, denoted by $f^{'}(x_0)$, signifies **the slope of the tangent line to the curve f(x) at point x_0,**, where the slope of a line represents the tangent of the angle made by that line with the Ox axis. Geometric interpretation of the derivative represented graphically:



Useful for Exercises

The following are considered known:

1. The equation of a line that passes through a point (x_0, y_0) and has slope m, is:

$$y - y_0 = m \cdot (x - x_0)$$

2. The slope **m** of a line of the form $y = m \cdot x + n$, represents the tangent of the angle made by the line with the Ox axis, therefore m=tg(α), where α represents the angle made by the line with the Ox axis.

3. The condition for two lines $y_1 = m_1 \cdot x + n_1$ and $y_2 = m_2 \cdot x + n_2$ to be parallel is that the lines have equal slopes, that is $m_1 = m_2$.

4. The condition for two lines $y_1 = m_1 \cdot x + n_1$ and $y_2 = m_2 \cdot x + n_2$ to be perpendicular is that $m_1 = -\frac{1}{m_2}$.

Exercises

1. Write the equation of the tangent to the curve f(x)=ln(x)+x^2-1 at point (e,e^2).

2. Determine a point on the curve y=$\frac{x-1}{x+1}$, x$\neq -1$, where the tangent is parallel to the line y=$\frac{x}{2}$.

3. Determine the constants α and β, such that the curves with equations:

$$y_1 = \alpha \cdot x^2 + \beta \cdot x + 2 \quad and \quad y_2 = \frac{x-a}{x}, x \neq 0$$

are tangent at point x=1, where $a \in R$ is a known value.

4. Write the equation of the normal to the parabola $y = x^2 - 4x + 5$ at the intersection points of this with the line x-y+1=0.

SHEET 31: Integrals

1. The Notion of Integral

The integral of function f(x) is denoted by $\int f(x) \cdot dx$. The notation $\int$ represents an *elongated* S, the letter S coming from the word Surface. It will be seen later that one of the most important applications of integrals is the calculation of areas.

2. Integral Table

$$
\begin{array}{lcl}
\int x^n \cdot dx & = & \frac{x^{n+1}}{n+1} \ + \ C, \ (n \neq -1) \\
\int a^x \cdot dx & = & \frac{a^x}{ln(a)} \ + \ C \\
\int \frac{1}{x} \cdot dx & = & ln|x| \ + \ C \\
\int \frac{1}{x+a} \cdot dx & = & ln|x+a| \ + \ C \\
\int \frac{1}{x^2-a^2} \cdot dx & = & \frac{1}{2 \cdot a} \cdot ln|\frac{x-a}{x+a}| \ + \ C \\
\int \frac{1}{x^2+a^2} \cdot dx & = & \frac{1}{a} \cdot arctg\frac{x}{a} \ + \ C \\
\int \frac{1}{\sqrt{x^2+a^2}} \cdot dx & = & ln|x+\sqrt{x^2+a^2}| \ + \ C \\
\int \frac{1}{\sqrt{x^2-a^2}} \cdot dx & = & ln|x+\sqrt{x^2-a^2}| \ + \ C \\
\int \frac{1}{\sqrt{a^2-x^2}} \cdot dx & = & arcsin\frac{x}{a} \ + \ C \\
\int sin(x) \cdot dx & = & -cos(x) \ + \ C \\
\int cos(x) \cdot dx & = & sin(x) \ + \ C \\
\int \frac{1}{cos^2(x)} \cdot dx & = & tg(x) \ + \ C \\
\int \frac{1}{sin^2(x)} \cdot dx & = & -ctg(x) \ + \ C \\
\int tg(x) \cdot dx & = & -ln|cos(x)| \ + \ C \\
\int ctg(x) \cdot dx & = & ln|sin(x)| \ + \ C
\end{array}
$$

3. Methods for Calculating Integrals

1) Substitution method: involves **solution recipes**
2) Integration by parts method:

$$\int f \cdot g^{'} \cdot dx = f \cdot g - \int f^{'} \cdot g \cdot dx$$

4. Definite Integrals. Leibniz-Newton Formula

- Let F(x) be the primitive of function f(x), that is $\int f(x) \cdot dx = F(x)$
- Leibniz-Newton Formula:

$$\int_a^b f(x) \cdot dx = F(x)|_a^b = F(b) - F(a)$$

5. Definite Integrals. Integration by Parts Method

$$\int_a^b f \cdot g^{'} \cdot dx = f \cdot g|_a^b - \int_a^b f^{'} \cdot g \cdot dx$$

6. Operations with Integrals

$\int (f+g) \cdot dx = \int f \cdot dx + \int g \cdot dx$

$\int (f-g) \cdot dx = \int f \cdot dx - \int g \cdot dx$

$\int k \cdot f \cdot dx = k \cdot \int f \cdot dx$, where k= constant

The previous formulas are valid both for indefinite integrals and for definite integrals.

7. Subtleties Regarding Integrals

- Riemann sums, applications to calculating sequence limits
- Darboux sums
- Integrable functions

8. Applications of Definite Integrals

Let f(x) be a function, the Cartesian system XOY, a line x=a and a line x=b

- Let S = area of the curve bounded by f(x), x=a, x=b and Ox

$$S = \int_a^b |f(x)| \cdot dx$$

- Let V = volume of the body obtained by rotating the curve bounded by f(x), x=a, x=b around the Ox axis.

$$V = \pi \cdot \int_a^b f^2(x) \cdot dx$$

- Let l_c = length of the curve f(x), between x=a and x=b

$$l_c = \int_a^b \sqrt{1 + (f'(x))^2} \cdot dx$$

- Let A = lateral area of the body obtained by rotating the curve bounded by f(x), x=a, x=b around the Ox axis.

$$A = 2 \cdot \pi \cdot \int_a^b |f(x)| \cdot \sqrt{1 + (f')^2} \cdot dx$$

- Let $G(x_g, y_g)$ be the center of gravity of the plate bounded by f(x), x=a, x=b, Ox. The coordinates of the center of gravity are:

$$x_g = \frac{\int_a^b x \cdot f(x) \cdot dx}{\int_a^b f(x) \cdot dx} \qquad y_g = \frac{\frac{1}{2} \cdot \int_a^b f^2(x) \cdot dx}{\int_a^b f(x) \cdot dx}$$

SHEET 32: Derivative Table and Integral Table

1. Derivative Table

Function	Derivative
k(=constant)	0
x	1
x^n	$n \cdot x^{n-1}$
$\sqrt{x}$	$\frac{1}{2 \cdot \sqrt{x}}$
$ln(x)$	$\frac{1}{x}$
a^x	$a^x \cdot ln(a)$
e^x	e^x
$sin(x)$	$cos(x)$
$cos(x)$	$-sin(x)$
$tg(x)$	$\frac{1}{cos(x)^2}$
$ctg(x)$	$-\frac{1}{sin(x)^2}$
$arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$arctg(x)$	$\frac{1}{1+x^2}$
$arcctg(x)$	$-\frac{1}{1+x^2}$

2. Integral Table

$$
\begin{array}{lcl}
\int x^n \cdot dx & = & \frac{x^{n+1}}{n+1} \; + \; C, \;\; (n \neq -1) \\
\int a^x \cdot dx & = & \frac{a^x}{ln(a)} \; + \; C \\
\int \frac{1}{x} \cdot dx & = & ln|x| \; + \; C \\
\int \frac{1}{x+a} \cdot dx & = & ln|x+a| \; + \; C \\
\int \frac{1}{x^2-a^2} \cdot dx & = & \frac{1}{2 \cdot a} \cdot ln|\frac{x-a}{x+a}| \; + \; C \\
\int \frac{1}{x^2+a^2} \cdot dx & = & \frac{1}{a} \cdot arctg\frac{x}{a} \; + \; C \\
\int \frac{1}{\sqrt{x^2+a^2}} \cdot dx & = & ln|x + \sqrt{x^2+a^2}| \; + \; C \\
\int \frac{1}{\sqrt{x^2-a^2}} \cdot dx & = & ln|x + \sqrt{x^2-a^2}| \; + \; C \\
\int \frac{1}{\sqrt{a^2-x^2}} \cdot dx & = & arcsin\frac{x}{a} \; + \; C \\
\int sin(x) \cdot dx & = & -cos(x) \; + \; C \\
\int cos(x) \cdot dx & = & sin(x) \; + \; C \\
\int \frac{1}{cos^2(x)} \cdot dx & = & tg(x) \; + \; C \\
\int \frac{1}{sin^2(x)} \cdot dx & = & -ctg(x) \; + \; C \\
\int tg(x) \cdot dx & = & -ln|cos(x)| \; + \; C \\
\int ctg(x) \cdot dx & = & ln|sin(x)| \; + \; C
\end{array}
$$

SHEET 33: Derivative and Integral Tables on One Page

1. Derivative Table

Function	Derivative
k(=constant)	0
x	1
x^n	$n \cdot x^{n-1}$
$\sqrt{x}$	$\frac{1}{2 \cdot \sqrt{x}}$
$ln(x)$	$\frac{1}{x}$
a^x	$a^x \cdot ln(a)$
e^x	e^x
$sin(x)$	$cos(x)$
$cos(x)$	$-sin(x)$
$tg(x)$	$\frac{1}{cos(x)^2}$
$ctg(x)$	$-\frac{1}{sin(x)^2}$
$arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$arctg(x)$	$\frac{1}{1+x^2}$
$arcctg(x)$	$-\frac{1}{1+x^2}$

2. Integral Table

$$
\begin{array}{lcl}
\int x^n \cdot dx & = & \frac{x^{n+1}}{n+1} \; + \; C, \;\; (n \neq -1) \\
\int a^x \cdot dx & = & \frac{a^x}{ln(a)} \; + \; C \\
\int \frac{1}{x} \cdot dx & = & ln|x| \; + \; C \\
\int \frac{1}{x+a} \cdot dx & = & ln|x+a| \; + \; C \\
\int \frac{1}{x^2-a^2} \cdot dx & = & \frac{1}{2 \cdot a} \cdot ln|\frac{x-a}{x+a}| \; + \; C \\
\int \frac{1}{x^2+a^2} \cdot dx & = & \frac{1}{a} \cdot arctg\frac{x}{a} \; + \; C \\
\int \frac{1}{\sqrt{x^2+a^2}} \cdot dx & = & ln|x + \sqrt{x^2+a^2}| \; + \; C \\
\int \frac{1}{\sqrt{x^2-a^2}} \cdot dx & = & ln|x + \sqrt{x^2-a^2}| \; + \; C \\
\int \frac{1}{\sqrt{a^2-x^2}} \cdot dx & = & arcsin\frac{x}{a} \; + \; C \\
\int sin(x) \cdot dx & = & -cos(x) \; + \; C \\
\int cos(x) \cdot dx & = & sin(x) \; + \; C \\
\int \frac{1}{cos^2(x)} \cdot dx & = & tg(x) \; + \; C \\
\int \frac{1}{sin^2(x)} \cdot dx & = & -ctg(x) \; + \; C \\
\int tg(x) \cdot dx & = & -ln|cos(x)| \; + \; C \\
\int ctg(x) \cdot dx & = & ln|sin(x)| \; + \; C
\end{array}
$$

This sheet was intended to contain in the most compact way possible the formulas that are frequently used in mathematical analysis.

In order to achieve this goal, the following recapitulations are considered appropriate:

Recapitulation 1: *Main Formulas for Derivatives*

1) Operations with derivatives:

$(f+g)^{'} = f^{'} + g^{'}$

$(f-g)^{'} = f^{'} - g^{'}$

$(f \cdot g)^{'} = f^{'} \cdot g + f \cdot g^{'}$

$(k \cdot f)^{'} = k \cdot f^{'}$, where k=constant

$(\frac{f}{g})^{'} = \frac{f^{'} \cdot g - f \cdot g^{'}}{g^2}$

2) Advanced $(f^g)^{'} = f^g \cdot (g^{'} \cdot ln(f) + g \cdot \frac{f^{'}}{f})$

3) Differentiation of composite functions: $f(u(v(x)))^{'} = f^{'}(u(v(x))) \cdot u^{'}(v(x)) \cdot v^{'}(x)$

Recapitulation 2: *Main Formulas for Integrals*

1) Substitution method: involves **solution recipes**

2) Integration by parts method:

$$\int f \cdot g^{'} \cdot dx = f \cdot g - \int f^{'} \cdot g \cdot dx$$

3) Operations with integrals:

$\int (f+g) \cdot dx = \int f \cdot dx + \int g \cdot dx$

$\int (f-g) \cdot dx = \int f \cdot dx - \int g \cdot dx$

$\int k \cdot f \cdot dx = k \cdot \int f \cdot dx$, where k= constant

4) Leibniz-Newton Formula:

$$\int_a^b f(x) \cdot dx = F(x)|_a^b = F(b) - F(a)$$

5) Area calculation:

Let S = area of the curve bounded by f(x), x=a, x=b and Ox

$$S = \int_a^b |f(x)| \cdot dx$$

Group

Let A be a non-empty set and an operation $*$

For example A=R and $*$: R x R $->$ R, where x$*$y=x+y-xy

(A,$*$) is a **group** if the conditions of stable part, associativity, neutral element, symmetric element are satisfied. If in addition commutativity is also satisfied, the group is called commutative group or abelian group.

1. Stable part:

For any x $\in$ A and y $\in$ A $\rightarrow$ x$*$y $\in$ A

2. Associativity:

For any x,y,z $\in$ A, we have x$*$(y$*$z)=(x$*$y)$*$z

3. Neutral element:

For any x $\in$ A, there exists an element e $\in$ A such that x$*$e=e$*$x=x

4. Symmetric element:

For any x $\in$ A, there exists an element x' $\in$ A such that x$*$x'=x'$*$x=e

5. Commutativity:

For any x,y $\in$ A, x$*$y=y$*$x

Examples of groups:

a) (R,+) Neutral element e=0. Symmetric for example for x=7, is x'=-7

b) (R\{0},.) Neutral element e=1. Symmetric for example for x=7, is x'=$\frac{1}{7}$

Monoid

Let A be a non-empty set and an operation $*$

(A,$*$) is a **monoid** if the conditions of associativity and neutral element are satisfied, that is:

1. Associativity:

For any x,y,z $\in$ A, we have x$*$(y$*$z)=(x$*$y)$*$z

2. Neutral element:

For any x $\in$ A, there exists an element e $\in$ A such that x$*$e=e$*$x=x

Ring

Let A be a non-empty set and two operations $*$ and $\circ$

(A,$*$,$\circ$) is a **ring** if the following conditions are satisfied:

1. (A,$*$) is an abelian group, that is

(A,$*$) = stable part

(A,$*$) = associativity

(A,$*$) = neutral element

(A,$*$) = symmetric element

(A,$*$) = commutativity

2. (A,$\circ$) is a monoid, that is

(A,$\circ$) = associativity

(A,$\circ$) = neutral element

3. Operation $\circ$ (the one with the monoid) **is distributive with respect to** $*$ (the one with the group) **both on the left and on the right**, that is:

a) x$\circ$(y$*$z)=(x$\circ$y)$*$(x$\circ$z) (left distributivity of $\circ$ with respect to $*$)

b) (y$*$z)$\circ$x=(y$\circ$x)$*$(z$\circ$x) (right distributivity of $\circ$ with respect to $*$)

Example of ring:

(R,+,$\cdot$) Addition plays the role of $*$ and multiplication plays the role of $\circ$. It is observed that the conditions of distributivity of multiplication with respect to addition are verified:

a) x$\cdot$(y+z)=(x$\cdot$y)+(x$\cdot$z) (left distributivity of multiplication with respect to addition)

b) (y+z)$\cdot$x=(y$\cdot$x)+(z$\cdot$x) (right distributivity of multiplication with respect to addition)

Notes

1. A **commutative ring** is called a ring in which operation $\circ$ is commutative.

2. A **field** is called a ring in which any element different from the neutral element of $*$ (the one with the group) is invertible. If operation $\circ$ (the one with the monoid) is commutative, then the field is called **commutative field**

Isomorphism

Let (A,$*$), (B,$\circ$) and a function f:A$\rightarrow$ B

1. It is said that we have a **morphism** from (A,$*$) to (B,$\circ$) through function f(x), if the following relation is satisfied:

$$f(x * y) = f(x) \circ f(y)$$

2. If in addition function f(x) is bijective, it is said that we have **isomorphism**. Therefore

isomorphism=morphism + f(x) bijective

Recall that a function is bijective if it is both injective and surjective.

EPILOGUE

You have completed the journey from Worksheet 1 to Worksheet 34. If you worked through this material seriously, you now possess something rare: a solid, rigorous foundation in mathematics that will serve you for life.

This epilogue addresses three important topics: what was deliberately excluded from this book, the philosophy behind its compact design, and a final word about courage.

Chapters Not Included in This Work

The purpose of this book was to synthesize the essential working tools of high school and early university mathematics.

The primary objective was for this work to be SHORT.
For this reason, only subjects considered strictly necessary were treated.

Chapters were not included if they met one or more of the following characteristics:

- Some are too easy and assumed already known
- Others were already covered in middle school
- Some are used less frequently in practice
- Others are too difficult for the intended scope
- Some are not required by standard examination syllabi
- Others appear very rarely or not at all in actual examinations
- And so forth

The focus was relentless: **include only what is essential**. Everything else, no matter how interesting, was deliberately excluded to keep this book compact and usable.

The Compact Edition Philosophy

This book is the **distilled, compact version** of a much larger work (approximately three times this length). The comprehensive version contains additional examples, supplementary exercises, extended discussions, and deeper theoretical treatments.

However, that comprehensive version would be **overwhelming for most users**. After 45 years of teaching, one lesson is crystal clear: students benefit more from mastering essentials than from being exposed to everything.

This compact edition represents a deliberate choice: **depth over breadth, mastery over coverage**.

If you work through this book systematically and master its contents, you will experience a **complete transformation** in your mathematical abilities. You will possess:

- Rock-solid foundations in core mathematics
- The ability to think rigorously and systematically
- Confidence to tackle advanced mathematics independently
- A framework that makes further learning vastly easier

This compact book contains everything you need to build that foundation. The comprehensive version exists for those who want more — but it is not necessary. This version is **sufficient and complete**.

Availability and Future Editions

This book is currently available in the following languages:

- **Romanian** (the original)
- **English** (this edition)
- **Spanish**

A translation into **French** is planned, so that hardworking students in that linguistic community can also access this resource.

For updates on translations and additional resources, visit: **rjurcone.net** or contact: **jurcone@yahoo.com**

A Final Word: On Courage

The word "courage" comes from the Latin *cor*, meaning "heart." To have courage means to act **from the heart**, even in the presence of fear.

If you are a young person today, you live in an age of unprecedented anxiety. Social media amplifies comparison and insecurity. Educational systems often prioritize performance metrics over genuine learning. The pressure to succeed can feel crushing, and the fear of failure paralyzing.

Let me tell you something important: **mathematical ability is not primarily about innate talent**. It is about **courage** — the courage to sit down with a difficult problem even when you don't immediately see the solution. The courage to persist when you feel stupid. The courage to try again after making mistakes.

This book was written by someone who learned mathematics **completely self-taught** — alone, from books, without tutors or external help. There were countless moments of confusion, frustration, and doubt. What made the difference was not brilliance, but **courage to continue**.

You have that same courage within you. The anxiety you feel is not a sign of inadequacy — it is a sign that you care deeply about succeeding. That care, channeled into persistent effort, is more powerful than any innate talent.

Here is what you need to know:

- **Confusion is normal.** Every mathematician experiences it constantly. It is not a sign of failure; it is a sign you are learning.
- **Mistakes are essential.** You learn more from working through errors than from getting things right immediately.
- **Slow progress is still progress.** Some students grasp concepts quickly; others need more time. Both paths lead to mastery if you persist.
- **You are not competing with others.** Your only competition is with yesterday's version of yourself. Are you better today than yesterday? That is success.
- **Hard work beats talent when talent doesn't work hard.** Consistent effort, even in small amounts, compounds over time into extraordinary results.

This book is the fruit of decades of teaching experience, designed specifically to help you learn independently. The quality is there. The methodology is proven.

What remains is your part: the courage to pick up this tool and use it, day after day, even when it feels difficult.

Mathematics is not a barrier designed to keep you out. It is a language — and like any language, it can be learned by anyone willing to practice consistently. You do not need to be a genius. You need to be **courageous** and **persistent**.

I wish you perseverance and success.

With respect and encouragement,

The Author